European Proceedings of
Social and Behavioural Sciences
EpSBS
www.europeanproceedings.com
e-ISSN: 2357-1330

DOI: 10.15405/epsbs.2020.06.50

ERD 2019
Education, Reflection, Development, Seventh Edition

STRUCTURES OF RINGS OF INTEGERS NUMBERS, ISOMORPHIC BETWEEN THEM

Teodor-Dumitru Vălcan (a)*
*Corresponding author

(a) Didactics of Exact Sciences Department, Babes-Bolyai University, Cluj-Napoca, Romania, tdvalcan@yahoo.ca

Abstract

Often solving a problem / exercise in a certain algebraic structure is quite difficult. That is why it is sometimes necessary to transfer the respective problem / exercise into an isomorphic structure with the given one and where it can be solved / studied more easily. But the problem of determining isomorphic algebraic structures at one time is quite difficult for students / teachers as well. In this paper we are proposing the construction of some rings isomorphic to the ring of integers numbers **Z**, on different subsets of the set **Z**. To begin with, we will see that if m is a integer number, then on the set of multiples of m, so on the set m·**Z** we can define such a structure. On the other hand, it is known that the set of natural numbers, **N**, does not form a ring structure with the usual operations of addition and multiplication of numbers. But, naturally, the question arises: On the set of **N** natural numbers, two internal operations can not be defined so that they give **N** an isomorphic ring structure with **Z**? We will see that the answer to this question is positive; we can define more such ring structures on any sets of natural numbers of the form m·**N**, where m is a natural number. In conclusion, we will show that for any natural number m, the sets m·**N** and m·**Z** can become isomorphic commutative rings with the ring **Z**.

Keywords: Mathematics, didactics of mathematics, integer number, commutative ring, isomorphism.

https://doi.org/10.15405/epsbs.2020.06.50
Corresponding Author: Teodor-Dumitru Vălcan
Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2357-1330

1. Introduction

As I have shown in Vălcan (2013), Mathematics is an object / discipline of education that is studied throughout schooling. Due to the complexity and open nature of Mathematics, its study cannot end at any level of learning. The multiple transformations that society has, the implications of Mathematics in all economic and social spheres, impose as a stringent necessity the best mathematical training for every citizen. However, education through Mathematics, a component part of education, must be done in accordance with both the transformations in society and the systemic, interactive nature of the teaching principles. That is why it is necessary to integrate in a unitary conception the different ways of perfecting all components of the educational process, in interrelation and their reporting to the finality of the whole process (education). Therefore, the mathematical education should be improved in the context of the improvement of the other components of this process, especially of the related ones.

The ways of conceiving and presenting the scientific knowledge systems of Mathematics, or their components, included in the curricula and translated into textbooks, collections, teaching guides, etc., present numerous errors. This is due to the observance of some malfunctioning structures built through school curricula, the faulty way of presenting the components of this knowledge, objectives not centered on the major outcomes of mathematical education, and more. The most important and frequent defect in the presentation of this knowledge is that it is not clear what elements of notional content are to be attributed to the study directly in the didactic process, i.e. the possibility of being more clear and logical to be presented by the teacher and more easily and more enjoyable to learn by pupils / students.

What is the possible solution to these fundamental problems of mathematical education, which in didactic terms would be formulated as interrogations such as:

- Is it necessary and possible to „*form*" a new image about Mathematics - science?
- From what premises should this new image start?
- How would this new image be transposed into the mathematical education and what would it change in the course of the didactic process through which it manifests itself?
- It can be made ordering the knowledge of Mathematics, in particular of notional content, so that it is accepted by the community of teachers?
- How would you be motivated and helped pupils to acquire better and more enjoyable than this discipline so far, by such a way of putting it of the Mathematics Learning Problem?

We have customized these more general questions into a specific, more substantive set of our research:

- What is the composition and the possible structure of an „*informational matrix*" specific to the notional content of Mathematics, defining the study of this science from the gymnasium to the university level?
- How would this „*informational matrix*" transpire in school and university curricula and in the explicit and / or implicit discourse of teachers?
- What element of this structure is essential in the internal reconstruction, after a new didactic logic, of this school discipline?

Here, we have outlined, so, a first problem to be researched, that of the scientific content of Mathematics studied in the education system, more specifically the search and characterization of the

https://doi.org/10.15405/epsbs.2020.06.50
Corresponding Author: Teodor-Dumitru Vălcan
Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2357-1330

essential element of the notional content, whose understanding depends on the appropriation of the other content components, as well as their application (Astolfi & Develey, 1989).

Because the whole mathematical notional content presented in school is organized algebraically structural, in fact, it is about forming a new image of the mathematical concepts of isomorphic algebraic structures by:

- discarding the influences of an exclusive materialistic philosophy;
- their study by applying the new psychological theories to the formation of mathematical notions;
- the formation of a new didactic logic around these concepts, because:
    - is important in forming an image of reality,
    - has a fundamental position in the construction of the knowledge system of Mathematics,
    - has a justification role in the formation of scientific / mathematical language,
    - is subject / object of knowledge and criterion in compliance the logic of science in general, and the logic of Mathematics in particular,
    - interdisciplinary implications which he has in acquisition scientific concepts in other areas are essential.

2. Problem Statement

Analyzing students' results in written, olympiad or admission contests, you find it easy to conclude that they have some difficulty in understanding some components of mathematical notional content. In this paper we will present ways of detecting the nature and causes of difficulties encountered by students in the teaching - learning of Mathematics, with increased emphasis on isomorphic algebraic structures.

Not a few times students encounter difficulties in learning these notions due to at least the following causes:

- the deficiencies of the analytical programs in force;
- poor presentation of notional content in textbooks;
- lips of problem collections, appropriate to the respective component (s) of the notional content;
- poor teacher training;
- poor presentation of the scientific content by teachers in class (Vălcan, 1997).

Concerning the theme suggested by the title of this paper, it should be noted that, so far, disparate articles were written only on certain components of the scientific content, in which their epistemic character prevailed. It was found that the components of the notional content were not grouped, by domains and / or themes, thus integrated into a knowledge system, much less by groups of related topics.

In general, teachers form their own opinions on how to teach scientific content, based on more or less well-founded experiences, and having as sources of information only school programs and textbooks. The number of studies emphasizing the methodological character of the scientific content processing is far too small compared to the many solutions offered by this mode of treatment, which is also based on the difficulties faced by students in the learning process. The causes that generate these difficulties, their nature, have been found so far, empirically, only from the observations made at student exams and / or their written works. The proposed solutions were the result of more theoretical analyzes. Attempts to solve concrete, following detailed pedagogical experiences, were very rare. We should also mention here

https://doi.org/10.15405/epsbs.2020.06.50
Corresponding Author: Teodor-Dumitru Vălcan
Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2357-1330

that most of the studies so far in this field, from Piaget (1970) to Gagné (1975) and the team led by Ausubel (1968), have dealt with the issue of acquiring notions by subjects in the context of research laboratory, and not by the pupils, in the didactic context, as the teaching - learning process takes place. Also, the development of didactic materials, programs and / or textbooks was moreover the result of some theoretical studies; generally, these were developed on the basis of experimental research as follows: theoretical documentation in cognitive psychology + pilot experimental research (theoretical treatment) + development of solutions (concretization on support materials).

It is noted, the need to investigate the difficulties faced by students in the context of the systemic approach of mathematical notional content, as well as its processing, in order to have a more comprehensive view on it.

A study of the difficulties encountered by students in the Mathematics training process must take into account the whole complexity of this process. The causes of difficulties can be found at the level of any component of the learning process. From this perspective we can identify:

- difficulties due to notional content,
- difficulties due to a misconception about the finality of the training process,
- difficulties due to the strategy used by teachers during teaching hours,
- difficulties due to poor assessment of pupils' knowledge,
- difficulties due to the communication between the teacher and the student (Vălcan, 1997).

It is noted that while the first category contains objective causes related to the specificity of Mathematics as a science and discipline of education, the last four categories of difficulties presented above highlight their possible causes in the study of Mathematics, factors related to human components (teacher - student), of the educational process.

Any investigation of the difficulties faced by students in teaching - learning mathematics, should lead to the discovery of answers at least to the following questions:

- to what extent did the mathematical concepts have been made wrong?
- how did these students not understand these components?
- why it is necessary to change the behavior of teachers and pupils in teaching - learning Mathematics?
- what would this change be in the behavior of the teacher and student?

Here's what it is the context in which the theme of this work falls, starting from the ideas of Vălcan (2017) and recalled in Vălcan (2019).

3. Research Questions

In our research we will try to find answers to the following questions:

-There are structures of commutative ring defined on sets of integers and which are isomorphic to the commutative ring of integers, $(\mathbf{Z},+,\cdot)$?

-How can these structures be identified?

https://doi.org/10.15405/epsbs.2020.06.50
Corresponding Author: Teodor-Dumitru Vălcan
Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2357-1330

4. Purpose of the Study

Therefore, we answered the two questions in Paragraph 3. Thus, for any number m, $p\in\mathbf{N}^*$ there are two pairs of laws of internal composition on the sets $m\cdot\mathbf{Z}$ and $p\cdot\mathbf{N}$, let's say „$\oplus$" and „$\otimes$", respectively „Δ" and „$\perp$", so that $(m\cdot\mathbf{Z},\oplus,\otimes)$ and $(p\cdot\mathbf{N},\Delta,\perp)$ become commutative rings isomorphic to the ring $(\mathbf{Z},+,\cdot)$.

Concretely, on the set of integers multiples of 3, $3\cdot\mathbf{Z}$ and on the set of natural multiples of 5, $5\cdot\mathbf{N}$ we can define two pairs of laws of internal composition so that let's say „$\oplus$" and „$\otimes$", respectively „Δ" and „$\perp$", so that $(3\cdot\mathbf{Z},\oplus,\otimes)$ and $(5\cdot\mathbf{N},\Delta,\perp)$ become commutative rings isomorphic to the ring $(\mathbf{Z},+,\cdot)$.

5. Research Methods

Let m be a natural number, n≥1, **N** – the set of natural numbers and the set:

$m\cdot\mathbf{Z}=\{m\cdot x \mid x\in\mathbf{Z}\}$,

of the integer multiples of m. Then the sets **N** and $m\cdot\mathbf{Z}$ are equipotent and write $\mathbf{N} \sim m\cdot\mathbf{Z}$ because there is a bijective map from **N** to $m\cdot\mathbf{Z}$:

$$\mathbf{N} \xrightarrow{g} \mathbf{Z} \xrightarrow{f} m\cdot\mathbf{Z},$$

where:

$$g(x)=\begin{cases}\dfrac{x}{2} & \text{, if x is even}\\ -\dfrac{x+1}{2} & \text{, if x is odd}\end{cases} \qquad \text{and} \qquad f(x)=m\cdot x,$$

are bijective functions. So,

$$h:\mathbf{N}\to m\cdot\mathbf{Z}, \qquad h=f\circ g,$$

that this:

$$h(x)=\begin{cases}\dfrac{m\cdot x}{2} & \text{, if x is even}\\ -\dfrac{m\cdot(x+1)}{2} & \text{, if x is odd}\end{cases},$$

is a bijection and:

$$h^{-1}: m\cdot\mathbf{Z}\to\mathbf{N}, \qquad h^{-1}(m\cdot x)=\begin{cases}2\cdot x & \text{, if } x\geq 0\\ -2\cdot x-1 & \text{, if } x<0\end{cases}.$$

The first fundamental result in this paragraph is:

***Theorem 5.1**: For every number $m\in\mathbf{N}^*$, there are two laws of internal composition, let's say „$\oplus$" and „$\otimes$", on the set $m\cdot\mathbf{Z}$, such that $(m\cdot\mathbf{Z},\oplus,\otimes)$ to become is a commutative ring isomorphic to the ring $(\mathbf{Z},+,\cdot)$.*

Proof: We transfer the ring structure from **Z** to $m\cdot\mathbf{Z}$. So, according to Vălcan (2017),.

$$m\cdot x\oplus m\cdot y=f(f^{-1}(m\cdot x)+f^{-1}(m\cdot y))=f(x+y)=m\cdot(x+y)=m\cdot x+m\cdot y, \tag{5.1}$$

$$m\cdot x\otimes m\cdot y=f(f^{-1}(m\cdot x)\cdot f^{-1}(m\cdot y))=f(x\cdot y)=m\cdot(x\cdot y), \tag{5.2}$$

$$e_{mZ}=f(0)=m\cdot 0=0 \qquad \text{and} \qquad -(m\cdot x)=m(-x), \tag{5.3}$$

and:

$$1_{mZ}=f(1)=m\cdot 1=m \qquad \text{and} \qquad (m\cdot x)^{-1}_{m\cdot Z}=f\left(\frac{1}{x}\right)=\frac{m}{x}. \tag{5.4}$$

Therefore, accroding to Vălcan (2017), $(m\cdot\mathbf{Z},\oplus,\otimes)$ is a commutative ring isomorphic to the ring $(\mathbf{Z},+,\cdot)$, by the function f, and the only invertible elements in the ring $m\cdot\mathbf{Z}$ are numbers: $m\cdot 1$ and $m\cdot(-1)$, ie m and –m.

Observe that, for every x, $y\in\mathbf{Z}$,

$$f(x+y)=m\cdot(x+y)=f(x)\oplus f(y) \quad \text{and} \quad f(x\cdot y)=m\cdot(x\cdot y)=f(x)\otimes f(y). \tag{5.5}$$

Remark 5.2: *For m=1, the above application f becomes the identity automorphism $1_{\mathbf{Z}}$ of the ring $(\mathbf{Z},+,\cdot)$.*

The next fundamental result is:

Theorem 5.3: *For every number $m\in\mathbf{N}^*$, there are two laws of internal composition, let's say „$*$" and „$\bullet$", on the set **N**, such that $(\mathbf{N},*,\bullet)$ to become is a commutative ring isomorphic to the ring $(m\cdot\mathbf{Z},\oplus,\otimes)$.*

Proof: We transfer the ring structure from $m\cdot\mathbf{Z}$ to **N**, using the function h^{-1}. Hence, according to Vălcan (2017), obtain the two composition laws „$*$" and „$\bullet$" on **N**. Let be x and y from **N**. For defining the law „$*$", we distinguish the following cases:

<u>Case 1</u>: x and y are even. Then:

$$x*y=h^{-1}(h(x)\oplus h(y))=h^{-1}\left(\frac{m\cdot x}{2}+\frac{m\cdot x}{2}\right)=h^{-1}\left(\frac{m\cdot(x+y)}{2}\right)=x+y. \tag{5.6}$$

<u>Case 2</u>: x and y are odd. Then:

$$x*y=h^{-1}(h(x)\oplus h(y))=h^{-1}\left(\frac{-m\cdot(x+1)}{2}+\frac{-m\cdot(y+1)}{2}\right)=h^{-1}\left(\frac{m\cdot(-x-y-2)}{2}\right)$$

$$=-2\cdot\left(\frac{m\cdot(-x-y-2)}{2}\right)-1=x+y+1. \tag{5.7}$$

<u>Case 3</u>: x is even and y is odd. Then:

$$x*y=h^{-1}(h(x)\oplus h(y))=h^{-1}\left(\frac{m\cdot x}{2}+\frac{-m\cdot(y+1)}{2}\right)=h^{-1}\left(\frac{m\cdot(-x-y-1)}{2}\right)$$

$$=\begin{cases}x-y-1, \text{ if } x\geq y+1\\ -x+y\ , \text{ if } x<y+1\end{cases}. \tag{5.8}$$

<u>Case 4</u>: x is odd and y is even. Then:

$$x*y=h^{-1}(h(x)\oplus h(y))=h^{-1}\left(\frac{-m\cdot(x+1)}{2}+\frac{m\cdot y}{2}\right)=h^{-1}\left(\frac{m\cdot(-x+y-1)}{2}\right)$$

$$=\begin{cases}-x+y-1, \text{ if } y\geq x+1\\ x-y \qquad , \text{ if } y<x+1\end{cases}. \tag{5.9}$$

Therefore, according to the equalities (5.6) - (5.9), for every x, $y\in\mathbf{N}$,

$$x*y=\begin{cases}x+y \quad , \text{ if x and y are even}\\ x+y+1, \text{ if x and y are odd}\\ x-y-1\ , \text{ if x is even, y is odd, and } x\geq y+1\\ -x+y\ , \text{ if x is even, y is odd, and } x<y+1\\ -x+y-1, \text{ if x is odd, y is even, and } y\geq x+1\\ x-y \quad , \text{ if x is odd, y is even, and } y<x+1\end{cases}. \tag{5.10}$$

Now, for defining the law „$\bullet$", we distinguish the following cases:

<u>Case 1</u>: x and y are even. Then:

https://doi.org/10.15405/epsbs.2020.06.50
Corresponding Author: Teodor-Dumitru Vălcan
Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2357-1330

$$x\bullet y=h^{-1}(h(x)\otimes h(y))=h^{-1}\left(\frac{m\cdot x}{2}\otimes\frac{m\cdot x}{2}\right)=h^{-1}\left(\frac{m\cdot(x+y)}{2}\right)=\frac{x\cdot y}{2}. \tag{5.11}$$

<u>Case 2</u>: x and y are odd. Then:

$$x\bullet y=h^{-1}(h(x)\otimes h(y))=h^{-1}\left(\frac{-m\cdot(x+1)}{2}\otimes\frac{-m\cdot(y+1)}{2}\right)=h^{-1}\left(\frac{m\cdot(x+1)\cdot(y+1)}{4}\right)$$

$$=\frac{(x+1)\cdot(y+1)}{2}. \tag{5.12}$$

<u>Case 3</u>: x is even and y is odd. Then:

$$x\bullet y=h^{-1}(h(x)\otimes h(y))=h^{-1}\left(\frac{m\cdot x}{2}\otimes\frac{-m\cdot(y+1)}{2}\right)=h^{-1}\left(\frac{m\cdot(-x)\cdot(y+1)}{4}\right)=\frac{x\cdot(y+1)}{2}-1. \tag{5.13}$$

<u>Case 4</u>: x is odd and y is even. Then:

$$x\bullet y=h^{-1}(h(x)\otimes h(y))=h^{-1}\left(\frac{-m\cdot(x+1)}{2}\otimes\frac{m\cdot y}{2}\right)=h^{-1}\left(\frac{m\cdot(-y)\cdot(y+1)}{4}\right)=\frac{y\cdot(x+1)}{2}-1. \tag{5.14}$$

Therefore, according to the equalities (5.11) and (5.14), for every x, y$\in$**N**,

$$x\bullet y=\begin{cases}\frac{x\cdot y}{2} & \text{, if x and y are even}\\ \frac{(x+1)\cdot(y+1)}{2} & \text{, if x and y are odd}\\ \frac{x\cdot(y+1)}{2}-1 & \text{, if x is even and y is odd}\\ \frac{y\cdot(x+1)}{2}-1 & \text{, if x is odd and y is even}\end{cases}. \tag{5.15}$$

On the other hand,

$$e_{\mathbf{N}}=h^{-1}(e_{m\cdot\mathbf{Z}})=h^{-1}(m\cdot 0)=0 \tag{5.16}$$

and

$$-x_{\mathbf{N}}=h^{-1}(-h(x))=\begin{cases}h^{-1}\left(-\frac{m\cdot x}{2}\right) & \text{, if x is even}\\ h^{-1}\left(\frac{m\cdot(x+1)}{2}\right) & \text{, if x is odd}\end{cases}=\begin{cases}x-1, \text{ if x is even}\\ x+1, \text{ if x is odd}\end{cases}, \tag{5.17}$$

and:

$$1_{\mathbf{N}}=h^{-1}(1_{m\cdot\mathbf{Z}})=h^{-1}(m\cdot 1)=2 \tag{5.18}$$

and

$$x^{-1}=h^{-1}\left(\frac{1}{m\cdot x}\right)=\begin{cases}2\cdot\frac{1}{x} & \text{, if } x\geq 0\\ -2\cdot\frac{1}{x}-1, & \text{if } x<0\end{cases}. \tag{5.19}$$

Therefore, according to Vălcan (2017), (**N**,$*$,$\bullet$) is a commutative ring isomorphic to the ring (m$\cdot$**Z**,$\oplus$,$\otimes$), by bijective function h^{-1}, and the only invertible elements in the ring **N** are:

$$1=h^{-1}(-m)=h^{-1}(m\cdot(-1)) \quad \text{and} \quad 2=h^{-1}(m)=h^{-1}(m\cdot 1). \tag{5.20}$$

Now let's show that, indeed, the function h^{-1} is an isomorphism between the two rings. For this, we first notice that, for every x, y$\in$**Z**,

https://doi.org/10.15405/epsbs.2020.06.50
Corresponding Author: Teodor-Dumitru Vălcan
Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2357-1330

$$h^{-1}(m\cdot x*m\cdot y)=h^{-1}(m\cdot(x+y))=\begin{cases}2\cdot(x+y) & ,\text{if } x+y\geq 0\\ -2\cdot(x+y)-1, & \text{if } x+y<0\end{cases} \tag{5.21}$$

and

$$h^{-1}(m\cdot x\bullet m\cdot y)=h^{-1}(m\cdot(x\cdot y))=\begin{cases}2\cdot(x\cdot y) & ,\text{if } x\cdot y\geq 0\\ -2\cdot(x\cdot y)-1, & \text{if } x\cdot y<0\end{cases}. \tag{5.22}$$

For the determination of $h^{-1}(m\cdot x)*h^{-1}(m\cdot y)$ we distinguish the following cases:

Case 1: $h^{-1}(m\cdot x)$ and $h^{-1}(m\cdot y)$ are even. Then:

$$h^{-1}(m\cdot x)*h^{-1}(m\cdot y)=m\cdot x+m\cdot y=m\cdot(x+y). \tag{5.23}$$

Case 2: $h^{-1}(m\cdot x)$ and $h^{-1}(m\cdot y)$ are odd. Then:

$$h^{-1}(m\cdot x)*h^{-1}(m\cdot y)=h^{-1}(m\cdot x)+h^{-1}(m\cdot y)+1=-2\cdot x-1-2\cdot y-1+1=-2\cdot(x+y)-1. \tag{5.24}$$

Case 3: $h^{-1}(m\cdot x)$ is even and $h^{-1}(m\cdot y)$ is odd. Then:

$$h^{-1}(m\cdot x)*h^{-1}(m\cdot y)=\begin{cases}h^{-1}(x)-h^{-1}(y)-1, & \text{if } h^{-1}(x)\geq h^{-1}(y)+1\\ -h^{-1}(x)+h^{-1}(y) & ,\text{if } h^{-1}(x)<h^{-1}(y)+1\end{cases}$$

$$=\begin{cases}2\cdot(x+y) & ,\text{if } x+y\geq 0\\ -2\cdot(x+y)-1, & \text{if } x+y<0\end{cases}. \tag{5.25}$$

Case 4: $h^{-1}(m\cdot x)$ is odd and $h^{-1}(m\cdot y)$ is even. Then:

$$h^{-1}(m\cdot x)*h^{-1}(m\cdot y)=\begin{cases}-h^{-1}(x)+h^{-1}(y)-1, & \text{if } h^{-1}(y)\geq h^{-1}(x)+1\\ h^{-1}(x)-h^{-1}(y) & ,\text{if } h^{-1}(y)<h^{-1}(x)+1\end{cases}$$

$$=\begin{cases}2\cdot(x+y) & ,\text{if } \cdot x+y\geq 0\\ -2\cdot(x+y)-1, & \text{if } x+y<0\end{cases}. \tag{5.26}$$

Therefore, according to the equalities (5.21) and (5.23) - (5.26), for every $x, y\in\mathbf{Z}$,

$$h^{-1}(m\cdot x*m\cdot y)=h^{-1}(m\cdot x)*h^{-1}(m\cdot y). \tag{5.27}$$

Now, for the determination of $h^{-1}(m\cdot x)\bullet h^{-1}(m\cdot y)$ we distinguish the following cases:

Case 1: $h^{-1}(m\cdot x)$ and $h^{-1}(m\cdot y)$ are even. Then:

$$h^{-1}(m\cdot x)\bullet h^{-1}(m\cdot y)=\frac{h^{-1}(m\cdot x)\cdot h^{-1}(m\cdot y)}{2}=2\cdot(x\cdot y). \tag{5.28}$$

Case 2: $h^{-1}(m\cdot x)$ and $h^{-1}(m\cdot y)$ are odd. Then:

$$h^{-1}(m\cdot x)\bullet h^{-1}(m\cdot y)=\frac{(h^{-1}(x)+1)\cdot(h^{-1}(y)+1)}{2}=2\cdot(x\cdot y). \tag{5.29}$$

Case 3: $h^{-1}(m\cdot x)$ is even and $h^{-1}(m\cdot y)$ is odd. Then:

$$h^{-1}(m\cdot x)\bullet h^{-1}(m\cdot y)=\frac{h^{-1}(x)\cdot(h^{-1}(y)+1)}{2}-1=-2\cdot(x\cdot y)-1. \tag{5.30}$$

Case 4: $h^{-1}(m\cdot x)$ is odd and $h^{-1}(m\cdot y)$ is even. Then:

$$h^{-1}(m\cdot x)\bullet h^{-1}(m\cdot y)=\frac{h^{-1}(y)\cdot(h^{-1}(x)+1)}{2}-1=-2\cdot(x\cdot y)-1. \tag{5.31}$$

Therefore, according to the equalities (5.22) and (5.28) - (5.31), for every $x, y\in\mathbf{Z}$,

$$h^{-1}(m\cdot x\bullet m\cdot y)=h^{-1}(m\cdot x)\bullet h^{-1}(m\cdot y). \tag{5.32}$$

https://doi.org/10.15405/epsbs.2020.06.50
Corresponding Author: Teodor-Dumitru Vălcan
Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2357-1330

From the equalities (5.27) and (5.32) it follows that the function h^{-1} is an isomorphism of rings and thus the rings ($\mathbf{N}$,$*$,$\bullet$) and ($m\cdot\mathbf{Z}$,$\oplus$,$\otimes$) are isomorphic.

Remark 5.4: *Also, the ring ($\mathbf{N}$,$*$,$\bullet$) is isomorphic to the ring ($\mathbf{Z}$,+,$\cdot$), by isomorphism g.*

Remark 5.5**:** *For m=1, from Theorem 5.2, obtain the ring structure on* $\mathbf{N}$*, transferred from the ring ($\mathbf{Z}$,+,$\cdot$), which is the same as that transferred from the ring ($m\cdot\mathbf{Z}$,$\oplus$,$\otimes$), for every $m\in\mathbf{N}^*$.*

In fact, we can say that, for every m, n, p, q$\in\mathbf{N}^*$, we have the following commutative diagram, of commutative rings:

$$\begin{tikzcd}
\mathbf{Z} \arrow[r] \arrow[d] \arrow[dr] & m\cdot\mathbf{Z} \arrow[r] \arrow[d] \arrow[dl] \arrow[dr] & n\cdot\mathbf{Z} \arrow[d] \arrow[dl] \\
\mathbf{N} \arrow[r] & p\cdot\mathbf{N} \arrow[r] & q\cdot\mathbf{N},
\end{tikzcd} \tag{5.33}$$

where,

$u : m\cdot\mathbf{Z} \to m\cdot\mathbf{Z}$, $v : \mathbf{N} \to p\cdot\mathbf{N}$, $t : p\cdot\mathbf{N} \to q\cdot\mathbf{N}$, $w : m\cdot\mathbf{Z} \to p\cdot\mathbf{N}$,

are defined by:

- for every $x\in\mathbf{Z}$, $u(m\cdot x)=n\cdot x$,
- for every $x\in\mathbf{N}$, $v(x)=p\cdot x$,
- for every $x\in\mathbf{N}$, $t(p\cdot x)=q\cdot x$,
- for every $x\in\mathbf{Z}$, $w(m\cdot x)=p\cdot g^{-1}(x)$,

and,

$v\circ g^{-1} : \mathbf{Z} \to p\cdot\mathbf{N}$, $t\circ w : m\cdot\mathbf{Z} \to q\cdot\mathbf{N}$, $w\circ u^{-1} : n\cdot\mathbf{Z} \to p\cdot\mathbf{N}$,

are defined by:

- for every $x\in\mathbf{Z}$, $(v\circ g^{-1})(x)=v(g^{-1}(x))=\begin{cases} 2\cdot p\cdot x & ,\text{if } x\geq 0 \\ -p\cdot(2\cdot x+1), & \text{if } x<0 \end{cases}$, (5.34)
- for every $x\in\mathbf{Z}$, $(t\circ w)(m\cdot x)=t(w(m\cdot x))=t(p\cdot g^{-1}(x))=q\cdot g^{-1}(x)=\begin{cases} 2\cdot q\cdot x & ,\text{if } x\geq 0 \\ -q\cdot(2\cdot x+1), & \text{if } x<0 \end{cases}$, (5.35)
- for every $x\in\mathbf{Z}$, $(w\circ u^{-1})(n\cdot x)=w(u^{-1}(n\cdot x))=w(m\cdot x)=p\cdot g^{-1}(x)=\begin{cases} 2\cdot p\cdot x & ,\text{if } x\geq 0 \\ -p\cdot(2\cdot x+1), & \text{if } x<0 \end{cases}$. (5.36)

The last fundamental result of this paper is:

Theorem 5.6**:** *For every number $p\in\mathbf{N}^*$, there are two laws of internal composition, let's say „Δ" and „$\perp$", on the set $p\cdot\mathbf{N}$, such that ($p\cdot\mathbf{N}$,Δ,$\perp$) to become is a commutative ring isomorphic to the ring ($\mathbf{Z}$,+,$\cdot$).*

Proof**:** We transfer the ring structure from $\mathbf{Z}$ to $p\cdot\mathbf{N}$, using the bijection function:

$k=v\circ g^{-1} : \mathbf{Z} \to p\cdot\mathbf{N}$,

where, according to the equality (5.34),for every $x\in\mathbf{Z}$,

$$k(x)=(v\circ g^{-1})(x)=v(g^{-1}(x))=\begin{cases} 2\cdot p\cdot x & ,\text{if } x\geq 0 \\ -p\cdot(2\cdot x+1), & \text{if } x<0 \end{cases}, \tag{5.34′}$$

and,

$k^{-1}=g\circ v^{-1} : p\cdot\mathbf{N} \to \mathbf{Z}$,

where, for every $x\in\mathbf{N}$,

$$k^{-1}(p\cdot x)=g(x)=\begin{cases}\dfrac{x}{2} & \text{, if } x \text{ is even}\\ -\dfrac{x+1}{2} & \text{, if } x \text{ is odd}\end{cases}, \tag{5.37}$$

since,

$v^{-1} : p\cdot\mathbf{N} \to \mathbf{N}$,

where, for every $x\in\mathbf{N}$,

$v^{-1}(p\cdot x)=x$.

Hence, according to Vălcan (2017), obtain the two composition laws „Δ" and „⊥" on **N**. Let be x and y from **N** and let be:

$a=p\cdot x$ and $b=p\cdot y$

from $p\cdot\mathbf{N}$. For defining the law „Δ", we distinguish the following cases:

Case 1: x and y are even. Then:

$$a\Delta b=k(k^{-1}(p\cdot x)+k^{-1}(p\cdot y))=k\left(\frac{x}{2}+\frac{y}{2}\right)=k\left(\frac{x+y}{2}\right)=p\cdot(x+y)=a+b. \tag{5.38}$$

Case 2: x and y are odd. Then:

$$a\Delta b=k(k^{-1}(p\cdot x)+k^{-1}(p\cdot y))=k\left(\frac{-(x+1)}{2}+\frac{-(y+1)}{2}\right)=k\left(\frac{-x-y-2}{2}\right)$$

$$=-p\cdot\left(2\cdot\frac{-x-y-2}{2}+1\right)=p\cdot(x+y+1)=a+b+p. \tag{5.39}$$

Case 3: x is even and y is odd. Then:

$$a\Delta b=k(k^{-1}(p\cdot x)+k^{-1}(p\cdot y))=k\left(\frac{x}{2}+\frac{-(y+1)}{2}\right)=k\left(\frac{x-y-1}{2}\right)$$

$$=\begin{cases}p\cdot(x-y-1), \text{ if } x\ge y+1\\ p\cdot(-x+y) \quad, \text{ if } x<y+1\end{cases}=\begin{cases}a-b-p, \text{ if } a\ge b+p\\ -a+b \quad, \text{ if } a<b+p\end{cases}. \tag{5.40}$$

Case 4: x is odd and y is even. Then:

$$a\Delta b=k(k^{-1}(p\cdot x)+k^{-1}(p\cdot y))=k\left(\frac{-(x+1)}{2}+\frac{y}{2}\right)=k\left(\frac{-x+y-1}{2}\right)$$

$$=\begin{cases}p\cdot(-x+y-1), \text{ if } y\ge x+1\\ p\cdot(x-y) \qquad, \text{ if } y<x+1\end{cases}=\begin{cases}-a+b-p, \text{ if } b\ge a+p\\ a-b \qquad, \text{ if } b<a+p\end{cases}. \tag{5.41}$$

Therefore, according to the equalities (5.38) - (5.41), for every $a, b\in p\cdot\mathbf{N}$,

$a=p\cdot x$ and $b=p\cdot y$

with $x, y\in\mathbf{N}$:

$$a\Delta b=\begin{cases}a+b & \text{, if } x \text{ and } y \text{ are even}\\ a+b+p & \text{, if } x \text{ and } y \text{ are odd,}\\ a-b-p & \text{, if } x \text{ is even, } y \text{ is odd and } a\ge b+p\\ -a+b & \text{, if } x \text{ is even, } y \text{ is odd and } a<b+p\\ -a+b-p & \text{, if } x \text{ is odd, } y \text{ is even and } b\ge a+p\\ a-b & \text{, if } x \text{ is odd, } y \text{ is even and } b<a+p\end{cases}. \tag{5.42}$$

For defining the law „⊥", we distinguish the following cases:

Case 1: x and y are even. Then:

$$a\perp b=k(k^{-1}(p\cdot x)\cdot k^{-1}(y))=k\left(\frac{x}{2}\cdot\frac{y}{2}\right)=k\left(\frac{x\cdot y}{4}\right)=\frac{p\cdot(x\cdot y)}{2}=\frac{a\cdot b}{2\cdot p}. \quad (5.43)$$

Case 2: x and y are odd. Then:

$$a\perp b=k(k^{-1}(p\cdot x)+k^{-1}(p\cdot y))=k\left(\frac{-(x+1)}{2}\cdot\frac{-(y+1)}{2}\right)=k\left(\frac{(x+1)\cdot(y+1)}{4}\right)$$

$$=\frac{p\cdot(x+1)\cdot(y+1)}{2}=\frac{(a+p)\cdot(b+p)}{2\cdot p}. \quad (5.44)$$

Case 3: x is even and y is odd. Then:

$$a\perp b=k(k^{-1}(x)+k^{-1}(y))=k\left(\frac{p\cdot x}{2}\cdot\frac{-p\cdot(y+1)}{2}\right)=h^{-1}\left(\frac{(-x)\cdot(y+1)}{4}\right)$$

$$=\frac{p\cdot x\cdot(y+1)}{2}-p=\frac{a\cdot(b+p)}{2\cdot p}-p. \quad (5.45)$$

Case 4: x is odd and y is even. Then:

$$a\perp b=k(k^{-1}(p\cdot x)+k^{-1}(p\cdot y))=k\left(\frac{-(x+1)}{2}\cdot\frac{p\cdot y}{2}\right)=k\left(\frac{(-y)\cdot(x+1)}{4}\right)$$

$$=\frac{p\cdot y\cdot(x+1)}{2}-p=\frac{b\cdot(a+p)}{2\cdot p}-p. \quad (5.46)$$

Therefore, according to the equalities (5.43) - (5.46), for every $a, b\in p\cdot\mathbf{N}$,

$a=p\cdot x$ and $b=p\cdot y$

with $x, y\in\mathbf{N}$:

$$a\perp b=\begin{cases}\frac{a\cdot b}{2\cdot p} & \text{, if x and y are even}\\ \frac{(a+p)\cdot(b+p)}{2\cdot p} & \text{, if x and y are odd}\\ \frac{a\cdot(b+p)}{2\cdot p}-p & \text{, if x is even and y is odd}\\ \frac{b\cdot(a+p)}{2\cdot p}-p & \text{, if x is odd and y is even}\end{cases}. \quad (5.47)$$

On the other hand,

$$e_{p\cdot\mathbf{N}}=k(ez)=k(0)=p\cdot 0=0 \quad (5.48)$$

and

$$-a_{p\cdot\mathbf{N}}=k(-k^{-1}(p\cdot x))=\begin{cases}0 & \text{, if } x=0\\ p\cdot x-p, & \text{if x is even and } x\geq 2\\ p\cdot x+p, & \text{if x is odd}\end{cases}=\begin{cases}0 & \text{, if } a=0\\ a-p, & \text{if x is even and } x\geq 2\\ a+p, & \text{if x is odd}\end{cases}, \quad (5.49)$$

and:

$$1_{p\cdot\mathbf{N}}=k(1)=2\cdot p \quad (5.50)$$

and

$$a^{-1}_{p\cdot N}=k\left(\frac{1}{k^{-1}(a)}\right)=k\left(\frac{1}{k^{-1}(p\cdot x)}\right).$$

https://doi.org/10.15405/epsbs.2020.06.50
Corresponding Author: Teodor-Dumitru Vălcan
Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2357-1330

But,

$$\frac{1}{k^{-1}(p\cdot x)}=\begin{cases}\dfrac{2}{x} & \text{, if x is even}\\ -\dfrac{2}{x+1} & \text{, if x is odd}\end{cases}.$$

So,

$$\frac{1}{k^{-1}(p\cdot x)}\in\mathbf{Z} \qquad \text{if and only if} \qquad x\in\{1,2\}.$$

Therefore, according to Vălcan (2017), ($p\cdot\mathbf{N}$,Δ,⊥) is a commutative ring isomorphic to the ring ($\mathbf{Z}$,+,·), by bijective function k, and the only invertible elements in the ring **N** are:

$$p=k(-1) \qquad \text{and} \qquad 2\cdot p=k(1). \tag{5.51}$$

Now let's show that, indeed, the function k is an isomorphism between the two rings. For this, we first notice that, for every x, y∈**Z**,

$$k(x+y)=\begin{cases}2\cdot p\cdot(x+y) & \text{, if } x+y\geq 0\\ -2\cdot p\cdot(x+y)-p, & \text{if } x+y<0\end{cases} \tag{5.52}$$

and

$$k(x\cdot y)=\begin{cases}2\cdot p\cdot(x\cdot y) & \text{, if } x\cdot y\geq 0\\ -2\cdot p\cdot(x\cdot y)-p, & \text{if } x\cdot y<0\end{cases}. \tag{5.53}$$

Now we determine the value of k(x)Δk(y). According to the above, we obtain:

$$k(x)\Delta k(y)=\begin{cases}k(x)+k(y) & \text{, if } x\geq 0 \text{ andy } \geq 0\\ k(x)+k(y)+p, & \text{if } x<0 \text{ and } y<0,\\ k(x)-k(y)-p & \text{, if } x\geq 0,\ y<0 \text{ and } k(x)\geq k(y)+p\\ -k(x)+k(y) & \text{, if } x\geq 0,\ y<0 \text{ and } k(x)<k(y)+p\\ -k(x)+k(y)-p, & \text{if } x<0,\ y\geq 0 \text{ and } k(y)\geq k(x)+p\\ k(x)-k(y) & \text{, if } x<0,\ y\geq 0 \text{ and } k(y)<k(x)+p\end{cases}$$

$$=\begin{cases}2\cdot p\cdot(x+y) & \text{, if } x\geq 0 \text{ and } y\geq 0\\ -2\cdot p\cdot(x+y)-p, & \text{if } x<0 \text{ and } y<0,\\ 2\cdot p\cdot(x+y) & \text{, if } x\geq 0,\ y<0 \text{ and } x+y\geq 0\\ -2\cdot p\cdot(x+y)-p & \text{, if } x\geq 0,\ y<0 \text{ and } x+y<0\\ 2\cdot p\cdot(x+y) & \text{, if } x<0,\ y\geq 0 \text{ and } x+y\geq 0\\ -2\cdot p\cdot(x+y)-p & \text{, if } x\geq 0,\ y<0 \text{ and } x+y<0\end{cases}=\begin{cases}2\cdot p\cdot(x+y) & \text{, if } x+y\geq 0\\ -2\cdot p\cdot(x+y)-p, & \text{if } x+y<0\end{cases}. \tag{5.54}$$

Therefore, according to the equalities (5.52) and (5.54), for every x, y∈**Z**,

$$k(x+y)=k(x)\Delta k(y). \tag{5.55}$$

Now we determine the value of k(x)⊥k(y). According to the above, we obtain:

$$k(x)\perp k(y)=\begin{cases}\dfrac{k(x)\cdot k(y)}{2\cdot p} & \text{, if x and y are even}\\ \dfrac{(k(x)+p)\cdot(k(y)+p)}{2\cdot p} & \text{, if x and y are odd}\\ \dfrac{k(x)\cdot(k(y)+p)}{2\cdot p} & \text{, if x is even and y is odd}\\ \dfrac{k(y)\cdot(k(x)+p)}{2\cdot p} & \text{, if x is odd and y is even}\end{cases}$$

https://doi.org/10.15405/epsbs.2020.06.50
Corresponding Author: Teodor-Dumitru Vălcan
Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2357-1330

$$=\begin{cases} 2\cdot p\cdot(x\cdot y) & ,\text{if } x\geq 0 \text{ and } y\geq 0 \\ 2\cdot p\cdot(x\cdot y) & ,\text{if } x<0 \text{ and } y<0 \\ -2\cdot p\cdot(x\cdot y)-p, & \text{if } x\geq 0 \text{ and } y<0 \\ -2\cdot p\cdot(x\cdot y)-p, & \text{if } x<0 \text{ and } y\geq 0 \end{cases} = \begin{cases} 2\cdot p\cdot(x\cdot y) & ,\text{if } x\cdot y\geq 0 \\ -2\cdot p\cdot(x\cdot y)-p, & \text{if } x\cdot y<0 \end{cases}. \quad (5.56)$$

Therefore, according to the equalities (5.53) and (5.56), for every x, $y\in\mathbf{Z}$,

$$k(x\cdot y)=k(x)\perp k(y). \quad (5.57)$$

From the equalities (5.55) and (5.57) it follows that the function k is an isomorphism of rings and thus the rings ($p\cdot\mathbf{N},\Delta,\perp$) and ($\mathbf{Z},+,\cdot$) are isomorphic.

At the end of this paragraph, two further remarks are required:

Remark 5.7*:** *For p=1, from Theorem 5.6, obtain the ring structure on* ***N*, transferred from the ring (*$\mathbf{Z},+,\cdot$*), which is the same as that obtained in Remark 5.5.*

***Remark 5.8*:** *For every $p\in\mathbf{N}^*$, the two internal operations, „Δ" and „⊥", on the set $p\cdot\mathbf{N}$, such that ($p\cdot\mathbf{N},\Delta,\perp$) to become a commutative ring can also be transferred from the ring ($m\cdot\mathbf{Z},\oplus,\otimes$), via the w function. Thus these two rings ($p\cdot\mathbf{N},\Delta,\perp$) and ($m\cdot\mathbf{Z},\oplus,\otimes$) are isomorphic.*

6. Findings

Therefore, we answered the two questions in Paragraph 3. Thus, for any number m, $p\in\mathbf{N}^*$ there are two pairs of laws of internal composition on the sets $m\cdot\mathbf{Z}$ and $p\cdot\mathbf{N}$, let's say „⊕" and „⊗", respectively „Δ" and „⊥", so that ($m\cdot\mathbf{Z},\oplus,\otimes$) and ($p\cdot\mathbf{N},\Delta,\perp$) become commutative rings isomorphic to the ring ($\mathbf{Z},+,\cdot$).

Concretely, on the set of integers multiples of 3, $3\cdot\mathbf{Z}$ and on the set of natural multiples of 5, $5\cdot\mathbf{N}$ we can define two pairs of laws of internal composition so that let's say „⊕" and „⊗", respectively „Δ" and „⊥", so that ($3\cdot\mathbf{Z},\oplus,\otimes$) and ($5\cdot\mathbf{N},\Delta,\perp$) become commutative rings isomorphic to the ring ($\mathbf{Z},+,\cdot$).

7. Conclusion

It is known that, for every $n\in\mathbf{N}^*$, the set of matrices of order n:

$L_n=\{x\cdot I_n \mid x\in\mathbf{Z}\}\subset\mathcal{M}_n(\mathbf{Z})$,

together with the usual addition and multiplication of the matrices, form a (commutative) ring isomorphic to the ring ($\mathbf{Z},+,\cdot$) – the verification is immediate. In conclusion, concidering those proven in Vălcan (2017) and those mentioned above, we can say that any integer number (in particular, any natural number) can be represented as an element of any set of numbers: $m\cdot\mathbf{Z}$, and $p\cdot\mathbf{N}$, with m and $p\in\mathbf{N}^*$, but also as a matrix of L_n.

More than that, $x\cdot I_n\in L_n$, is not the only matrix representation of the integer number x; it can easily prove that the sets of matrices:

$H=\{A_{l,n}(x) \mid x\in\mathbf{Z}\}\subset\mathcal{M}_l(\mathbf{Z})$ and $K=\{A_{n,r}(x) \mid x\in\mathbf{Z}\}\subset\mathcal{M}_n(\mathbf{Z})$

where, for every $x\in\mathbf{Z}$, $A_{l,n}(x)=(a_{ij})\in\mathcal{M}_n(\mathbf{Z})$ is the matrix that has on line l all the elements equal to x and in the rest all are equal to 0, that is:

$$a_{ij}=\begin{cases} x, \text{ if } i = 1 \text{ and } j = \overline{1,n} \\ 0, \text{ if } i \neq 1 \text{ and } j = \overline{1,n} \end{cases},$$

and $A_{n,r}(x)=(a'_{ij})\in\mathcal{M}_n(\mathbf{Z})$ is the matrix that has on the column r all the elements equal to x and in the rest all are equal to 0, that is:

$$a'_{ij}=\begin{cases} x, \text{ if } j = r \text{ and } i = \overline{1,n} \\ 0, \text{ if } j \neq r \text{ and } i = \overline{1,n} \end{cases},$$

can be equipped with two laws of internal composition, say „•" and „⊥", respectively „∗" and „Λ" so that (H,•,⊥) and (K,∗,Λ) becomes commutative rings isomorphic to the ring ($\mathbf{Z}$,+,·).

Finally, for any a, b$\in\mathbf{Z}$, the functions:

$f_1 : \mathbf{Z} \to \mathbf{Z}$ and $f_2 : \mathbf{Z} \to \mathbf{Z}$,

where, for every $x\in\mathbf{Z}$,

$f_1(x)=x+a$ and $f_2(x)=-x+b$,

are bijective, and:

$f_1^{-1} : \mathbf{Z} \to \mathbf{Z}$ and $f_2^{-1} : \mathbf{Z} \to \mathbf{Z}$,

where, for every $x\in\mathbf{Z}$,

$f_1^{-1}(x)=x-a$ and $f_2^{-1}(x)=-x+b$.

According to Vălcan (2017) there are two pairs of laws of internal composition, let's say „♦" and „◊", respectively „♥" and „♠" such that ($\mathbf{Z}$,♦,◊) and ($\mathbf{Z}$,♥,♠) to become two commutative rings isomorhpic to the ring ($\mathbf{Z}$,+,·). These four laws of internal composition on the set $\mathbf{Z}$ are: for every x, y$\in\mathbf{Z}$,

$x♦y=f_1(f_1^{-1}(x)+f_1^{-1}(y))=f_1(x+y-2\cdot a)=x+y-a$,

$x◊y=f_1(f_1^{-1}(x)\cdot f_1^{-1}(y))=f_1((x-a)\cdot(y-a))=(x-a)\cdot(y-a)+a=x\cdot y-a\cdot x-a\cdot y+a^2+a$,

$x♥y=f_2(f_2^{-1}(x)+f_2^{-1}(y))=f_2(-x-y+2\cdot b)=x+y-b$,

$x◊y=f_2(f_2^{-1}(x)\cdot f_2^{-1}(y))=f_2((-x+b)\cdot(-y+b))=-(-x+b)\cdot(-y+b)+b=-x\cdot y+b\cdot x+b\cdot y-a^2+b$.

In conclusion, the diagram (5.33) can be supplemented with these four commutative rings isomorphic to the ring ($\mathbf{Z}$,+,·) and with any other ring of this type.

References

Astolfi, J. P., & Develey, M., (1989), *La didactique des sciences*. Paris: PUF.

Ausubel, D. P., (1968), *Educational psychology, A cognitive view*. New-York: Holt, Rinehart and Winston, Inc.

Gagné, M. R., (1975), *Condiţiile învăţării* [*Learning conditions*]. Bucureşti: Editura Didactică şi Pedagogică.

Piaget, J., (1970). *Psihologia copilului* [*Child Psychology*]. Bucureşti: Editura Didactică şi Pedagogică.

Vălcan, D., (1997). Investigarea dificultăţilor întâmpinate de elevi în învăţarea unor componente ale conţinutului noţional la disciplinele de ştiinţele naturii [Investigating the difficulties faced by students in learning some components of the notional content in the disciplines of natural sciences], *Lucrările Conferinţei Anuale a Societăţii de Ştiinţe Matematice din România*, *II*, 157-181.

Vălcan, D., (2013). *Didactica Matematicii* [*Didactics of Mathematics*]. București: Editura Matrix Rom.

https://doi.org/10.15405/epsbs.2020.06.50
Corresponding Author: Teodor-Dumitru Vălcan
Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2357-1330

Vălcan, D., (2017). An approach from the perspective triple of a problem of Algebra. *International Journal of Innovative and Applied Research*, 6(2), 5-32.

Vălcan, D., (2019). A Field Of Function Isomorphic To The field Of Real Numbers. *The European Proceedings of Social & Behavioural Sciences EpSBS*, *ERD 2018*, 419-429.