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PROPERTIES OF INDICES OF THE DISPROPORTIONALITY FOR INTEGER DIVISIONS

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Abstract

The condition of indivisibility in the problem of dividing a finite number of indivisible goods basically disables a division that is proportional to the claims of its participants. As a result, a need emerges to measure the disproportionality of a specific division, and in particular to answer the question what are the conditions for a given function to be a disproportionality index. The aim of the study is to define a minimum set of criteria defining the index of disproportionality taking into account the condition of the indivisibility of goods. Moreover to describe the selected properties of indices of disproportionality in order to facilitate the interpretation of the results of their use. In addition to the review and some form of classification of the functioning indices of disproportionality, a set of criteria is given which the disproportionality index should meet, in which the Pigou-Dalton transfer principle plays a key role. Most of the considerations are of a theoretical nature based on a mathematical apparatus. In selected cases, simulations were carried out on specific sets of divisions. A new index of disproportionality was defined, which is normalized weighted arithmetic mean in relation to the representation of claims regarding goods. Indices based on the vector norm order the set of divisions practically in the same way. Of the existing indices, the Least square index, the Sine index and the Saint-Lague index have outstanding properties, although the Saint-Lague index does not meet the normalized criterion.

Keywords: Disproportionality indices, Pigou-Dalton transfer principle.

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1. Introduction

Distributing a finite set of identical and indivisible goods among participants with specific entitlements so as to best represent the structure of claims is an issue typically examined in the context of allocating mandates in legislative bodies among political parties based on ballots. In this paper, to emphasize a more general approach, the terms: a participant in allocation or an agent will be used as a replacement for political party, just as entitlements for ballots, a good for a mandate or a parliamentary seat.

The consequence of indivisibility in the issue of the separation of the finite number of indivisible goods is often the inability of division in proportion to the claims of its participants. In existing studies addressing the measurement of disproportionality, the condition of integer division is usually omitted.

1.1. Notation

Let us consider a set of possible divisions of a finite number s of goods among n agents subject to $n < s$ where $n, s \in \mathbb{N}$ and satisfying the assumptions that an agent with greater entitlements cannot get less than an agent with smaller entitlements. Let $\boldsymbol{P} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \dots, \boldsymbol{p}_n)$ denote the vector of claims, i.e. the vector of weights corresponding to entitlements of agents, whereas one can assume without loss of generality that this vector is nondecreasing, i.e. if $i < j$ then $\boldsymbol{p}_i \le \boldsymbol{p}_j$, and $p = \sum_{i=1}^{n} \boldsymbol{p}_i$. Let $\boldsymbol{S} = (\boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_n)$ denote the vector defining the number of goods allocated to individual agents such that $s = \sum_{i=1}^{n} \boldsymbol{s}_i$ The set of these divisions will be denoted by $D_{\mathbb{R}_+}(n, s)$. If we also assume that the goods are indivisible, we get the set of integer divisions denoted by $D_{\mathbb{N}}(n, s)$ and $D(n, s)$. This is a set of n-element nondecreasing vectors whose elements are nonnegative integers summing up to s. Obviously, $D(n, s) \subset D_{\mathbb{R}_+}(n, s)$. $P = (p_1, p_2, \dots, p_n)$ is the vector of shares in claims, i.e. $P = \frac{1}{p}\boldsymbol{P}$, and $S = (s_1, s_2, \dots, s_n) = \frac{1}{s}\boldsymbol{S}$ is the vector of shares in goods. The coefficient $\frac{\boldsymbol{p}_i}{\boldsymbol{s}_i}$ is the representativeness of claims with respect to allocated goods for the i^{th} agent, and $\frac{\boldsymbol{s}_i}{\boldsymbol{p}_i}$ its reciprocal is the representativeness of goods with respect to claims for the i^{th} agent.

2. Problem Statement

2.1. Proportional division

The division $\boldsymbol{S}$ is called proportional with respect to $\boldsymbol{P}$ if its structure S corresponds to structure of P, i.e. if $S = P$ or if $\boldsymbol{S} = \alpha\boldsymbol{P}$, where $\alpha = s/p$. A proportional division is characterized by the fact that if the vector of representativeness is a constant sequence, then it holds $\frac{\boldsymbol{p}_1}{\boldsymbol{s}_1} = \frac{\boldsymbol{p}_2}{\boldsymbol{s}_2} = \dots = \frac{\boldsymbol{p}_n}{\boldsymbol{s}_n} = \frac{p}{s}$.

Definitely as a rule proportional divisions are not integer. For example, if one allocates eight goods among three agents, then $D(3,8) = \{(0,0,8), (0,1,7), (0,2,6), (0,3,5), (0,4,4), (1,1,6), (1,2,5), (1,3,4), (2,2,4), (2,3,3)\}$. If claims of agents are represented by the vector $\boldsymbol{P} = (2,3,6)$, then quota division is the vector $\frac{8}{11}(2,3,6) \notin D(3,8)$. Thus, the choice is ambiguous. Each vector of (1,2,5), (1,3,4) and (2,2,4) can be regarded as “near” to proportional division, because the term ‘near’ is not precise[1].

[1] In real applications, where no ambiguity is allowed, the problem of choice of proportional division is solved by arbitrary algorithm to determine such a division. The description of algorithms based on divisor method along with the cases of their applications can be found in Pukelsheim (2017) and Colomer (2016).

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There is also a terminological problem. If one determines a vector of integers that is nearest to a proportional division, based on additional premises, it is typically called a proportional division. Hence, to avoid a terminological unambiguity, a proportional division that is not integer division will be called strictly proportional or quota division[2], and will be denoted by $\boldsymbol{S}^*$.

The lack of proportionality, i.e. disproportionality, can also result from additional conditions imposed on a division. For example, a rule can be introduced that to get a good, an agent must claim more than an expected threshold, or conversely, when it is expected that each participant in division must be allocated a certain number of goods, irrespective of the quantity of their claims.

2.2. The concept of index of disproportionality

The index of disproportionality, also termed as disproportionality measure, index of distortion, and deviation from proportionality, is a function allowing to delineate how much a given division is different from proportional division. A discourse continues onto the meaning of such an index, its desired properties, and determining which functions currently employed to this end are such an index, and which are not.

One can state that application of such index has a twofold purpose. First, to evaluate how much the division under study is different from the quota division, i.e. to what degree it deviates from proportionality. Second, to establish an order in the set of integer deviations, i.e. ranking them so as to determine which one in any pair of divisions is nearer to a proportional division.

In 1991, Gallagher analyzed the properties of six indices used at that time, while in 1994, Monroe extended the investigation to nineteen indices. Similar studies were conducted by Taagepera and Grofman (2003) and by Karpov (2008), whereas their sets of indices do not overlap. Koppel and Diskin (2009), similarly as Chessa and Fragnelli (2012), put forward additional indices. Certainly, these studies do not exhaust the topic, and the number of potential indices of disproportionality is infinite. The compilations focus on determining which index under study is better, i.e. satisfies more required criteria, whereas the choice of criteria in different papers is unlike.

Taagepera and Grofman (2003) building on the work by Monroe (1994), identified 12 different criteria: seven are theoretical and five are practical, such as ease of computations and unambiguity of results. Karpov (2008) considers five different properties axiomatically approached, whereas Koppel and Diskin (2009) examine eight. Some of these criteria are put forward in all mentioned reviews, they only differ in terms of wording. There are also some criteria proposed in just one paper, and some whose wording or the usage in compilation raise doubts.

A given function can be considered the index of proportionality if it has the property that it attains the minimum for a quota division, and its greater values indicate more disparity between the given division and the quota division. It should also meet the classical criteria typical for the index of inequality.

One can also adopt a criterion to unambiguously determine a division without use of divisor methods, such as minimizing the sum of squares (see Łyko and Rudek 2013).

[2] This term stems from the concept of quota. A quota of the i^{th} agent is called the value $\boldsymbol{s}_i^* = \boldsymbol{p}_i \frac{s}{p}$.

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2.3. Criteria of the index of disproportionality

In this paper, the index of disproportionality will be the function $f\colon \boldsymbol{S} \times \boldsymbol{P} \to [0,\infty)$, where $\boldsymbol{S} \in D(n,k)$ and $\boldsymbol{P} \in \mathbb{R}_{+}^{n}$, satisfying the following criteria:

1. The Pigou-Dalton's transfer principle. Each transfer of a good from the agent with greater representativeness to the agent with smaller representativeness, without reordering representativeness, reduces the value of the function.
   $\mathrm{S}, \mathrm{S}' \in \mathrm{D}(\mathrm{n},\mathrm{k})$ such that $\boldsymbol{s}_i' = \boldsymbol{s}_i + \epsilon, \boldsymbol{s}_j' = \boldsymbol{s}_j - \epsilon$, where $\epsilon > 0$ and $\boldsymbol{s}_k' = \boldsymbol{s}_k$ for $\mathrm{k} \neq \mathrm{i},\mathrm{j}$. If $\frac{\mathrm{s_i}}{\mathrm{p_i}} < \frac{\mathrm{s_j}}{\mathrm{p_j}}$ and $\frac{\mathrm{s_i}+\epsilon}{\mathrm{p_i}} < \frac{\mathrm{s_j}\text{-}\epsilon}{\mathrm{p_j}}$ then $\mathrm{f}(\mathrm{S},\mathrm{P}) > \mathrm{f}(\mathrm{S}',\mathrm{P})$.
2. Unbiasedness. Any permutation of division participants does not influence the value of the function.
   $f(\boldsymbol{S},\boldsymbol{P}) = f(\pi(\boldsymbol{S}),\pi(\boldsymbol{P}))$ where $\pi(\boldsymbol{S})$ permutation of the vector $\boldsymbol{S}$.
3. Scale invariance. Multiplying the vector S or P by any positive number does not change the value of the function.
   $f(\lambda\boldsymbol{S},\boldsymbol{P}) = f(\boldsymbol{S},\lambda\boldsymbol{P}) = f(\boldsymbol{S},\boldsymbol{P})$ where $\lambda > 0$.

Moreover, the following criterion is proposed:

4. Normalized. The values of the function belong to the $[0,1]$ interval.
   $0 \leq \mathrm{f}(\mathrm{S},\mathrm{P}) \leq 1$.

2.4. Remarks on criteria

Some authors extend the canon of indices of inequality by decomposability criterion, postulating correlation between the general value of the index for the entire set and the values of indices evaluated for arbitrarily selected subsets. This condition is invalid when the domain is the set of determined divisions. The values of indices are determined by the structure of the set of integer divisions; whether this criterion is satisfied depends on division, not on the function. Modifying the number of either agents or goods implies the change of the domain.

The Pigou-Dalton transfer principle is a standard condition that should be satisfied by the index of inequality. Taagepera and Grofman (2003) define the Dalton's principle of transfers[3] as follows: "when a seat is transferred from a richer component to a poorer one, the disproportionality index should decrease". Such interpretation needs a more precise definition of terms 'richer' and 'poorer' in the context of proportional division. Is richer the one that has got more apart from considering due claims or the one with greater representativeness? Being aware of this issue, Taagepera and Grofman examine both possibilities, however, as demonstrated in Goldenberg and Fisher (2017), their classification according to this principle includes some errors. Koppel and Diskin (2009) provide an analogue of this condition defining a segment between the structure of claims P and the structure of goods S. If the vector of claims S' belongs to this segment, then $f(S',P) < f(S,P)$ holds. Such statement is too restrictive, because the structure of a set of integer divisions leads to a case when for every division $\boldsymbol{S}$ there is no division whose structure belongs to

[3] Hugh Dalton was the first to formalize the principle in 1920, however at the same time Arthur Cecil Pigou also researched into it, hence referring to the principle in various papers is somewhat inconsistent.

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the segment between the structure of claims and the structure of this division. If the condition of belonging to the segment is replaced by the condition that if the distance between P and S' is smaller than the distance between P and S, then $f(S', P) < f(S, P)$, and the condition equivalent to criterion 1 is obtained. Karpov (2008) defines the principle of transfers as follows: "If we transfer seats from an overrepresented party to an underrepresented party the value of the index should not increase". Similarly as understanding the statement given in Taagepera and Grofman (2003), one should also realize that this transfer does not result in the modified ranking of representativeness. With this adjustment, the statement is equivalent to criterion 1 in a weak form.

In criterion 1, instead of $\boldsymbol{S}$ and $\boldsymbol{P}$ one can consider S and P, and instead of the function $f(\boldsymbol{S}, \boldsymbol{P})$ – the function $f(S, P)$. This way of presentation is valuable, because the function $f(S, P)$ satisfies the criterion 3 without any further adjustments. Many authors presume from the outset that they do not deal with the set of integer divisions, but with n-element vectors whose elements sum up to one. Doing so, we should bear in mind that the domain with respect to the first argument is a discrete set, and that a possible transfer or a difference between vectors is a multiple $1/n$.

The function meeting criterion 1 has a uniquely defined minimum for proportional division: $f(\boldsymbol{S}, \boldsymbol{P}) \geq f(\alpha\boldsymbol{P}, \boldsymbol{P})$ and $f(\boldsymbol{S}, \boldsymbol{P}) = f(\alpha\boldsymbol{P}, \boldsymbol{P}) \Leftrightarrow \boldsymbol{S} = \alpha\boldsymbol{P}$ where $\alpha = s/p$. Alternatively: $f(S, P) \geq f(P, P)$ and $f(S, P) = f(P, P) \Leftrightarrow S = P$. One can also consider a weaker variant of this criterion. If $\frac{\boldsymbol{s}_i}{\boldsymbol{p}_i} < \frac{\boldsymbol{s}_j}{\boldsymbol{p}_j}$ and $\frac{\boldsymbol{s}_i + \epsilon}{\boldsymbol{p}_i} < \frac{\boldsymbol{s}_j - \epsilon}{\boldsymbol{p}_j}$ then $f(\boldsymbol{S}, \boldsymbol{P}) \geq f(\boldsymbol{S}', \boldsymbol{P})$. In this case the minimum is not necessarily unique.

Criteria 2 and 3 correspond to standard conditions of the inequality index. Any function that can be represented in the form of sum, product, maximum or minimum of functions depending only on single values, such as $f(\boldsymbol{S}, \boldsymbol{P}) = \sum_{i=1}^{n} g(\boldsymbol{s}_i, \boldsymbol{p}_i)$ or $f(\boldsymbol{S}, \boldsymbol{P}) = \max_{i=1,\dots,n} g(\boldsymbol{s}_i, \boldsymbol{p}_i)$, satisfies criterion 2. To satisfy criterion 3, the division can be replaced by the structure of the division: use $f(S, P)$ instead of $f(\boldsymbol{S}, \boldsymbol{P})$. One can also proceed in such a way that the value of f is respectively divided or multiplied by scaling constants s and p. In other words, criteria 2 and 3 are not key in the context of proportionality or a lack of it, but in the context of desired properties of the index.

Criterion 4 is not always required (and often is not satisfied). It serves rather as a reference point when analyzing the properties of indices.

2.5. Examples of indices of disproportionality

One approach to define the index of disproportionality is the use of the concept of the vector norm. Any function $f(S, P) = c\|S - P\|$, where $c > 0$ is a normalizing constant, meets criteria 1–3. The Rae, Duncan or Grofman indices are scaled city block norms of the vector of the difference between the structures of claims and goods: $f(S, P) = c\|S - P\|_1 = c\sum_{i=1}^{n}|s_i - p_i|$, where $c = \frac{1}{n}$ for the Rae index, $c = \frac{1}{2}$ for the Duncan index, and $c = \sum_{i=1}^{n} p_i^2$ for the Grofman index. The Gallagher index, often called the Least squares index, , uses the Euclidean norm: $f_{Lsq}(S, P) = \frac{1}{\sqrt{2}}\|S - P\|_2 = \sqrt{\frac{1}{2}\sum_{i=1}^{n}(s_i - p_i)^2}$, the Maximum Deviation index $f_{MD}(S, P) = \|S - P\|_\infty = \max_{i\in\{1,\dots,n\}}|s_i - p_i|$ is the Chebyshev norm of the vector that is the difference between the structures of vectors of goods and of claims.

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The other approach exploits the fact that the vector of representativeness for the quota division has to be a constant sequence, and the more discrepancy of the vector of representativeness for a given distribution from the constant sequence, the more disproportionality of the system. This inequality can be measured by well-known indices used to measure inequality such as the Gini index, the Atkinson index and the index of the generalized entropy, or by functions directly examining the structure of representativeness, such as the d'Hondt index: $f_H(S,P) = \max_{i\in\{1,\dots,n\}} \frac{s_i}{p_i}$, or the Saint-Laguë index, that is equivalent to the chi-square index: $f_{SL}(S,P) = \sum_{i=1}^{n} p_i\left(\frac{s_i}{p_i}-1\right)^2 = \sum_{i=1}^{n}\frac{(s_i-p_i)^2}{p_i}$ (see Renwick, 2015; Goldenberg & Fisher ,2017).

Other interesting indices of disproportionality proposed lately include functions based on the cosine of an angle between the two vectors. The function $\mathrm{Cos}(S,P) = \frac{\sum_{i=1}^{n} s_i p_i}{\sqrt{\sum_{i=1}^{n} s_i^2}\sqrt{\sum_{i=1}^{n} p_i^2}}$ (see Koppel & Diskin, 2009; Chessa & Fragnelli, 2012; Colignatus, 2017) has the followimg properties: $0 \le \mathrm{Cos}(S,P) \le 1$, $\mathrm{Cos}(S,P) = 1 \Leftrightarrow S = P$, $\mathrm{Cos}(S,P) = 0 \Leftrightarrow s_1p_1 = \dots = s_np_n = 0$ and $\|S-P\| > \|S'-P\| \Rightarrow \mathrm{Cos}(S,P) < \mathrm{Cos}(S',P)$. Therefore, the functions such as $1-\mathrm{Cos}(S,P)$ or $\mathrm{Sin}(S,P) = \sqrt{1-\mathrm{Cos}(S,P)^2}$ are indices of disproportionality.

There are also some functions satisfying criteria 1–3 that require imposing additional restrictions on the set of divisions and are dedicated to specific applications. An example of such index is the normalized weighted arithmetic mean with respect to the vector of representativeness of entitlements to goods (see 6.1). A condition is also considered that the vector of representativeness $\boldsymbol{P/S} = \left(\frac{p_1}{s_1},\frac{p_2}{s_2},\dots,\frac{p_n}{s_n}\right)$ generates a nondecreasing sequence, i.e. when the division $\boldsymbol{S}$ is integer and degressively proportional. This distribution is intermediate between the equal division and the proportional division. The degree of its proportionality versus the degree of its equality can be evaluated by the Dniestrzanski index given by formula: $f_{Dn}(S,P) = \sum_{i=1}^{n}(s_i-p_i)^2 \,/\, \sum_{i=1}^{n}\left(s_i-\frac{1}{n}\right)^2$ (Dniestrzański, 2014).

3. Research Questions

The study aims to clarify:

- how to measure the resulting disproportionality and what properties the indices of disproportionality should have.
- how to take into account the condition of the division's disproportionality and what consequences it can generate.
- if it is desired that the properties of disproportionality indices are unambiguously interpreted and compared.

Undoubtedly, the practice of comparing the allocations of parliamentary seats based on the ballots in a given region in various periods by the same index is correct and necessary. But what is the interpretation of the outcome when the index of a given allocation is 20 percent? Does it mean that the division is "extremely disproportional" or, quite the opposite, rather near to a proportional division?

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4. Purpose of the Study

The purpose of this study is to present the key role of the Pigou-Dalton transfer principle in defining the disproportionality indices. The aim of the study is also to define a minimum set of criteria defining the index of disproportionality taking into account the condition of the indivisibility of goods. Moreover to describe the selected properties of indices of disproportionality in order to facilitate the interpretation of the results of their use.

5. Research Methods

Apart from defining the minimum set of criteria of the disproportionality indices (see 2.3 and 2.4), the properties of the indices are examined by analysing the values of the indices for the extreme divisions. Moreover, the simulations are carried out by calculating the value of selected indices on the set of all possible, acceptable divisions.

6. Findings

6.1. Normalized weighted mean

If a given function meets criterion 1 and is adequately construed to meet criterion 2, then meeting criterion 3 merely needs a suitable scaling. To illustrate this idea, a new index of disproportionality. is introduced below.

Assuming that each participant in division has to obtain at least one good, the vector of representativeness can be considered in the form $\boldsymbol{P/S} = \left(\frac{p_1}{s_1}, \frac{p_2}{s_2}, \dots, \frac{p_n}{s_n}\right)$. The term $\frac{p_i}{s_i}$ signifies the amount of claims per one good for the i^{th} agent. Is it possible that the function measuring the average amount of claims per one good for all agents can be a starting point to generate the index of disproportionality? Given the proportional division, these representations are the same and equal $\frac{p}{s}$. The arithmetic mean does not satisfy the Pigou-Dalton transfer principle, hence there are some divisions whose mean representativeness vector is smaller than $\frac{p}{s}$. It is the case when the number of agents with small and very small claims significantly exceeds the number of agents with great claims. Considering the structure of claims, we deal with the weighted arithmetic mean with respect to the vector of claims, i.e. the function

$$f_{WAM}(\boldsymbol{S},\boldsymbol{P}) = P \circ \frac{\boldsymbol{P}}{\boldsymbol{S}} = \sum_{i=1}^{n} \frac{\boldsymbol{p}_i}{p} \frac{\boldsymbol{p}_i}{\boldsymbol{s}_i} = \frac{1}{p} \sum_{i=1}^{n} \frac{\boldsymbol{p}_i^2}{\boldsymbol{s}_i}.$$

Lemma. Weighted arithmetic mean with respect to the vector of entitlements to goods satisfies criterion 1.

Proof. Let $\boldsymbol{S}$ and $\boldsymbol{S}'$ be different only at two coordinates i and j: $\boldsymbol{s}_i' = \boldsymbol{s}_i + \epsilon, \boldsymbol{s}_j' = \boldsymbol{s}_j - \epsilon$, where $\epsilon > 0$, whereas $\frac{s_i}{p_i} < \frac{s_j}{p_j}$ and $\frac{s_i+\epsilon}{p_i} < \frac{s_j-\epsilon}{p_j}$. Hence $\frac{s_i}{p_i}\frac{s_i+\epsilon}{p_i} < \frac{s_j}{p_j}\frac{s_j-\epsilon}{p_j} \Leftrightarrow \frac{p_i^2(\epsilon-s_i+s_i)}{s_i(s_i+\epsilon)} > \frac{p_j^2(\epsilon-s_j+s_j)}{s_j(s_j+\epsilon)} \Leftrightarrow \frac{p_i^2}{s_i^2} - \frac{p_i^2}{s_i+\epsilon} > \frac{p_j^2}{s_j^2-\epsilon} - \frac{p_j^2}{s_j} \Leftrightarrow \frac{p_i^2}{s_i^2} + \frac{p_j^2}{s_j} > \frac{p_i^2}{s_i+\epsilon} + \frac{p_j^2}{s_j^2-\epsilon} \Leftrightarrow f_{WM}(\boldsymbol{S},\boldsymbol{P}) > f_{WM}(\boldsymbol{S}',\boldsymbol{P})$.

Due to the construction of the function, an arbitrary permutation of vectors S and P does not affect its values, thus criterion 2 is met. To satisfy criterion 3, a proper normalization is needed.

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Theorem. Normalized weighted arithmetic mean with respect to the vector of representativeness of entitlements to goods is given by the formula

$$f_{NWAM}(\boldsymbol{S},\boldsymbol{P}) = \frac{s}{p^2}\left(\sum_{i=1}^{n}\frac{\boldsymbol{p}_i^2}{\boldsymbol{s}_i} - \frac{p^2}{s}\right)$$

or

$$f_{NWAM}(S,P) = \sum_{i=1}^{n}\frac{p_i^2}{s_i} - 1$$

meets criteria 1–3, thus it is the index of disproportionality.

Proof. $f_{WAM}\left(\frac{s}{p}\boldsymbol{P},\boldsymbol{P}\right) = f_{WAM}(P,P) = \frac{p}{s}$. Meeting criterion 1 signifies in particular that $f_{WM}(\boldsymbol{S},\boldsymbol{P}) \geq \frac{p}{s}$ and $f_{WM}(\boldsymbol{S},\boldsymbol{P}) = \frac{p}{s}$ if and only if $\boldsymbol{S} = \frac{s}{p}\boldsymbol{P}$. $f_{NWAM}(\lambda\boldsymbol{S},\boldsymbol{P}) = f_{NWAM}(\boldsymbol{S},\lambda\boldsymbol{P}) = f_{NWAM}(\boldsymbol{S},\boldsymbol{P})$ and $f_{NWAM}(\boldsymbol{S},\boldsymbol{P}) \geq 0$.

6.2. Analysis of extreme divisions

Due to criteria 2 and 3, one can limit the analysis to the sets of vectors satisfying the conditions $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} s_i = 1$. The first extreme case emerges when all goods are distributed to agents with zero claims, i.e. when: $P = (0, p_2, p_3, \dots, p_n)$ and $S = (1,0,\dots,0)$. The second case comes out when all goods are allocated to the agent with smallest positive claims, i.e. when $0 < p_1 \leq p_2 \leq \cdots \leq p_n$ and $S = (1,0,\dots,0)$. Both cases violate the rule that an agent with greater entitlements cannot get less than an agent with smaller claims. If this rule is respected, then $0 < p_1 \leq p_2 \leq \cdots \leq p_n$ and $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n$. In the set of divisions satisfying this condition, the most remote vectors are $A = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$ and vector $B = (0,0,\dots,0,1)$.

In the first case, the indices based on the vector of representativeness S/P have no value, because $\frac{s_1}{p_1}$ does not exist. The Saint-Laguë index in the second and third case can take arbitrarily great values. In case two $f_{SL}(S,P) = \frac{1}{p_1} - 2 + \sum_{i=1}^{n} p_i = \frac{1-p_1}{p_1} \geq n-1$, beaucose $0 < p_1 \leq \frac{1}{n}$. In case three, with $P = A$ and $S = B$ we get $f_{SL}(S,P) = \frac{n-1}{n}(0-1)^2 + \frac{1}{n}(n-1-1)^2 = n - 4 + \frac{4}{n}$. Likewise, infinitely great values can be taken by the d'Hondt index. In case two, $f_H(S,P) = \frac{1}{p_1}$, while in case three, $f_H(S,P) = n$. Thus, these indices do not satisfy criterion 4..

The Duncan index, the Least square index and the Maximum Deviation index are construed so as to yield one in the first case. In the second case, the Duncan index $f_D(S,P) = \frac{1}{2}(1 - p_1 + \sum_{i=2}^{n} p_i) = 1 - p_1$, and in the third case $f_D(S,P) = \frac{1}{2}\|A - B\|_1 = \frac{1}{2}\left(\frac{(n-1)}{n} + 1 - \frac{1}{n}\right) = 1 - \frac{1}{n}$. The Least square index in case two is given by $f_{Lsq}(S,P) = \sqrt{\frac{1}{2}(1 - 2p_1 + \sum_{i=1}^{n} p_i^2)} \leq \sqrt{1 - p_i}$, because if $\sum_{i=1}^{n} p_i = 1$, then $\sum_{i=1}^{n} p_i^2 \leq 1$. In case three, $f_{Lsq}(S,B) = \sqrt{\frac{n-1}{n^2} + \frac{(n-1)^2}{n^2}} = \sqrt{1 - \frac{1}{n}}$. The values of the Maximum Deviation index in case two and three are the same as the values of the Duncan index.

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The condition that the maximum value of the index, i.e. the case one, equals 1, is satisfied by the Sine index. In case two, $\mathrm{Cos}(S,P)=\frac{p_1}{\sqrt{\sum_{i=1}^{n}p_i^2}}$, in case three $\mathrm{Cos}(S,P)=\frac{\frac{1}{n}}{\sqrt{n\frac{1}{n^2}}}=\frac{1}{\sqrt{n}}$, hence in case two $\mathrm{Sin}(S,P)=\frac{\sqrt{\sum_{i=2}^{n}p_i^2}}{\sqrt{\sum_{i=1}^{n}p_i^2}}$, and in case three $\mathrm{Sin}(S,P)=\sqrt{1-\frac{1}{n}}$.

The Gini index handles all these three cases equally, because the vector of representativeness S/P is then a vector whose all coordinates, but one, are zeroes. In this case, the Gini index equals $1-\frac{1}{n}$.

6.3. Simulations

The interesting conclusions follow appropriate simulations evaluating the values of selected indices for all possible divisions that satisfy the condition stipulating that a greater agent does not get less than a smaller agent. Table 1 presents one example of simulation, whereas the presented conclusions reflect a greater number of simulations.

Table 01. Values of selected indices in the set of division of 9 goods among 3 agents with claims (3.3.4)

divisions			Lsq		Saint Laguë		Sin		Gini		MD		Duncan		d'Hondt -1	
3	3	4														
3	3	3	0,058	1	0,019	1	0,14	1	0,061	1	0,067	1	0,067	1	0,1111	1
2	3	4	0,068	2	0,028	2	0,159	2	0,083	2	0,078	2	0,078	2	0,1111	1
2	2	5	0,135	3	0,111	3	0,296	3	0,151	3	0,156	3	0,156	3	0,3889	3
1	4	4	0,171	4	0,201	5	0,379	5	0,25	5	0,189	4	0,189	4	0,4815	5
1	3	5	0,175	5	0,182	4	0,373	4	0,237	4	0,189	4	0,189	4	0,3889	3
1	2	6	0,238	6	0,304	6	0,468	6	0,311	6	0,267	6	0,267	6	0,6667	7
0	4	5	0,260	7	0,438	7	0,515	7	0,344	7	0,3	7	0,3	7	0,4815	5
0	3	6	0,285	8	0,481	8	0,537	8	0,4	9	0,3	7	0,3	7	0,6667	7
1	1	7	0,327	9	0,652	9	0,577	9	0,391	8	0,378	9	0,378	9	0,9444	9
0	2	7	0,346	10	0,664	10	0,599	10	0,483	10	0,378	9	0,378	9	0,9444	9
0	1	8	0,427	11	0,898	11	0,668	11	0,571	11	0,489	11	0,489	11	1,2222	11
0	0	9	0,520	12	1,5	12	0,728	12	0,667	12	0,6	11	0,6	12	1,5	12

The d'Hondt index is not respectively distinct, i.e. it combines different divisions, The probability that at least two divisions will have the same value almost indicates certainty. To a lesser degree, the Maximum Deviation index and the Duncan index also combine divisions. In case of the Least squares index, the Saint-Laguë index and the Sine index, the probability that at least two divisions will have the same value is very small.

The results of computations in case of the Maximum Deviation index and the Duncan index are in many cases similar. Rankings, i.e. the effects of ordering from the division with the smallest to the greatest value of an index, are similar in case of indices based on vector norm. They differ in case of the Least squares index, the Saint-Laguë index, the Sine index and the Gini index. The values of indices are fastest-increasing in case of the Sine index. The value of an index around 25 percent is a lot. In most cases of

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indices (except for the Sine, Maximum Deviation and Duncan indices), more than a half divisions have smaller values.

7. Conclusion

When examining the proportionality or its lack, the key criterion for the underlying function is to satisfy the Pigou-Dalton transfer principle. Proceeding to select a "fair" division, i.e. the division "nearest" to the proportional division from a given group of divisions by means of the index of dispsroportionality is a rational method. In case of doubts as to which index of disproportionality should be selected, one can apply several indices at the same time, whereas the d'Hondt index is not recommended, as well as simultaneous usage of the Least squares index with the Maximum Deviation and the Duncan indices. Among extant indices, the properties of the Least squares index, the Sine index and the Saint-Laguë index are eminent, but the Saint-Laguë index does not satisfy the normalized criterion.

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