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FROM INTEGRABLE FUNCTIONS TO LIMITS OF SEQUENCES

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Abstract

The formation and development of methodological competences to apply mathematical knowledge to solving exercises and problems must be one of the fundamental objectives of each Mathematics teacher. Unfortunately, this is very rare, precisely because professors do not have such competences. This paper aims to be an example of training and developing these competences for a very sensitive field: the calculation of the limits of sequences by special methods, more precisely with the help of definite integrals. It is known that certain limits of convergent sequences can be calculated using definite integrals. On the other hand, any function f integrable over an interval [a, b] induces a convergent sequence, that of the Riemann sums, which converges to the value of the integral of f on [a,b]. In this paper we show that in addition to this sequence, such a function f, which takes only positive values, may also lead to two more convergent sequences, and the calculation of their limits is also reduced to the calculation of the integral off over the interval [a, b]. The idea is that through the results and examples presented here to increase the sphere of the methodological competences of pupils / students to calculation of limits of convergent sequences.

Keywords: Integrated feature, convergent sequence, limit, inequalities. .

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1. Introduction

Mathematics is an object of education that is studied throughout schooling. Because of the complexity and open nature of Mathematics, its study can not end at any level of learning. The multiple transformations that society has, the implications of Mathematics in all economic and social spheres, impose as a stringent necessity the best possible mathematical training for every citizen.

Analyzing the results obtained by the pupils in the written, olympiad or admissions competitions, you find it easy to conclude that they have some difficulty in understanding some of the components of the notional content of Mathematics, especially in Mathematical Analysis, mostly due to of the faculty classroom teaching mode (Vălcan, 2013, p. 62).

Of course, by teaching Mathematics in school we now understand the transmission of mathematical knowledge and the presentation of working techniques with this knowledge.

Mathematical knowledge is condensed scientific information - in the form of: representations, notions, principles, mathematical sentences (remarks, propositions, lemmas, theorems, corollaries, consequences) and deduction rules, etc.

It is also known that Mathematical Analysis has many applications in science and technology. Therefore, a concern of Mathematics teachers should be to educate students a set of methodological competences to apply the theoretical knowledge in solving a wide range of types of exercises and problems.

This is why the present work comes to show a framework for the development of methodological competences to solve of differential calculus problems by means of integral calculus, issues very little known by teachers or pupils / students.

2. Problem Statement

It is known that there are convergent sequences whose limit can only be calculated with the help of the defined integral; for example, if:

$$a_n=\frac{1}{n}\cdot\left(\frac{1}{\sqrt{n^2+1^2}}+\frac{2}{\sqrt{n^2+2^2}}+\Lambda+\frac{n-1}{\sqrt{n^2+(n-1)^2}}+\frac{n}{\sqrt{n^2+n^2}}\right), \quad (1)$$

then, this sequence can be written like this:

$$a_n=\frac{1}{n}\cdot\sum_{k=1}^{n}\frac{\frac{k}{n}}{\sqrt{1+\left(\frac{k}{n}\right)^2}}. \quad (1')$$

Now, we consider the function:

$$f:[0,1]\to R, \qquad f(x)=\frac{x}{\sqrt{1+x^2}}. \quad (2)$$

This function is continuous on the interval [0,1], so it is also integrable on this interval. Therefore, according to Sirețchi (1985, p. 310), for every partition of the interval [0,1]:

$$\Delta_n=(0=x_0<x_1<x_2<\Lambda<x_n=1), \quad (3)$$

with:

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$$\lim_{n\to\infty} \|\Delta_n\|=0 \tag{4}$$

and any system of intermediate points:

$$\xi=(\xi_1,\xi_2,\xi,\Lambda\ ,\xi_n), \tag{5}$$

where, for any $k=\overline{1,n}$,

$$\xi_k\in[x_{k-1},x_k], \tag{6}$$

the sequence of the Riemann sums,

$$\sigma_{\Delta_n}(f,\xi)=\sum_{k=1}^{n}(x_k-x_{k-1})\cdot f(\xi_k) \tag{7}$$

is convergent and:

$$\lim_{n\to\infty}\sigma_{\Delta_n}(f,\xi)=\int_0^1 f(x)\cdot dx\,. \tag{8}$$

If:

$$\Delta_n=\left(0=\frac{0}{n}<\frac{1}{n}<\frac{2}{n}<\Lambda\ <\frac{n}{n}=1\right) \tag{9}$$

and:

$$\xi=\left(\frac{1}{n},\frac{2}{n},\Lambda\ ,\frac{n}{n}\right), \tag{10}$$

then, the partition Δ_n is of the form (3), the equality (4) is satisfied because:

$$\lim_{n\to\infty}\|\Delta_n\|=\lim_{n\to\infty}\frac{1}{n}=0, \tag{11}$$

the system ξ of the intermediate points is of the form (5), and belonging (6) she is also satisfied. On the other hand, according to the equality (7), the sequence $\sigma_{\Delta_n}(f,\xi)$ becomes:

$$\sigma_{\Delta_n}(f,\xi)=\frac{1}{n}\cdot\sum_{k=1}^{n}\frac{\frac{k}{n}}{\sqrt{1+\left(\frac{k}{n}\right)^2}}=a_n. \tag{12}$$

Therefore, according to the equalities (12) and (8), we have:

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\sigma_{\Delta_n}(f,\xi)=\int_0^1 f(x)\cdot dx=\int_0^1\frac{x}{\sqrt{1+x^2}}\cdot dx=\sqrt{1+x^2}\Big|_0^1=\sqrt{2}-1.$$

3. Research Questions

In our research we started from finding answers to the following questions:

-There are other types of sequences whose limit can be calculated using a definite integral?

-How can these sequences be identified?

-For the calculation of their limits we have only one method?

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4. Purpose of the Study

Further we put the problem in reverse, that is we will show that any positive function f, integrable on [0,1], lead always to training two types of convergent sequences, different from each other and different from the sequence (7), and calculating the limits of these new sequences is also reduced to the calculation of the integral of f on the interval [0,1].

5. Research Methods

In this paragraph we will present the main results that will lead to the above mentioned sequences. To begin with, we present the following technical result:

Lemma: *For any* $x\in[0,1]$*, the following inequalities hold*:

$$\frac{2\cdot x}{2+x}\leq \ln(1+x)\leq x; \tag{13}$$

respectively:

$$x-\frac{x^2}{2}\leq \ln(1+x)\leq x. \tag{14}$$

Proof**:** Let be:

$u:[0,1]\to\mathbf{R}$, $v:[0,1]\to\mathbf{R}$ and $w:[0,1]\to\mathbf{R}$

where, for every $x\in[0,1]$,

$u(x)=\ln(1+x)-\frac{2\cdot x}{2+x}$ $v(x)=\ln(1+x)-x+\frac{x^2}{2}$ and $w(x)=\ln(1+x)-x.$

Then:

$$u(0)=v(0)=w(0)=0 \tag{15}$$

and, for every $x\in(0,1)$,

$$u'(x)=\frac{x^2}{(1+x)\cdot(2+x)^2}\qquad v'(x)=\frac{x^2}{(1+x)^2}\qquad \text{and}\qquad w'(x)=-\frac{x}{(1+x)^2}. \tag{16}$$

From the equalities (16), according to Duca (2000, p. 261), it follows that, for any $x\in(0,1)$:

$u'(x)>0$ $v'(x)>0$ and $w'(x)<0$.

Hence, u and v are functions strictly increasing on [0,1], and w is a function strictly decreasing on [0,1]. In conclusion, according to the equalities (15), for any $x\in[0,1]$, we have the inequalities:

$w(x)\leq 0\leq u(x)$, respectively $w(x)\leq 0\leq v(x)$,

which is equivalent to the inegualities (13), respectively (14).

The main result of this paper is the following:

Theorem: *If* $f:[0,1]\to\mathbf{R}_+^*$ *is a integrable function, then*:

$$\lim_{n\to\infty}\frac{1}{n^2}\cdot\sum_{k=1}^{n}\frac{1}{\ln\left(1+\frac{\frac{1}{n}}{f\left(\frac{k}{n}\right)}\right)}=\int_0^1 f(x)\cdot dx\,. \tag{17}$$

and:

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$$\lim_{n\to\infty}\prod_{k=1}^{n}\left[1+\frac{1}{n}\cdot f\left(\frac{k}{n}\right)\right]=e^{\int_0^1 f(x)\cdot dx}. \tag{18}$$

***Proof*:** Let be:

$$x_n=\frac{1}{n^2}\cdot\sum_{k=1}^{n}\frac{1}{\ln\left(1+\dfrac{\frac{1}{n}}{f\left(\frac{k}{n}\right)}\right)}. \tag{19}$$

Since, according to the hypothesis,

$$\lim_{n\to\infty} n\cdot f\left(\frac{k}{n}\right)=+\infty,$$

It follows that, there is a $n_0\in\mathbf{N}$, such that, for any $n\in\mathbf{N}$, with $n>n_0$, and, for every $k=\overline{1,n}$,

$$0<\frac{1}{n\cdot f\left(\frac{k}{n}\right)}<1. \tag{20}$$

On the other hand, for every $x\in(0,1)$, from the inequalities (13), it follows that:

$$\frac{1}{x}\le\frac{1}{\ln(1+x)}\le\frac{1}{x}+\frac{1}{2}. \tag{13′}$$

Then according to the inequalities (13′) and (20), for every $n\in\mathbf{N}$, with $n>n_0$ and, for every $k=\overline{1,n}$,

$$n\cdot f\left(\frac{k}{n}\right)\le\frac{1}{\ln\left(1+\dfrac{\frac{1}{n}}{f\left(\frac{k}{n}\right)}\right)}\le n\cdot f\left(\frac{k}{n}\right)+\frac{1}{2}. \tag{21}$$

Summing up after $k=\overline{1,n}$, member with member, the inequalities (21), we obtain that:

$$n\cdot\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\le\sum_{k=1}^{n}\frac{1}{\ln\left(1+\dfrac{\frac{1}{n}}{f\left(\frac{k}{n}\right)}\right)}\le n\cdot\sum_{k=1}^{n}f\left(\frac{k}{n}\right)+\frac{n}{2}. \tag{22}$$

By dividing the inequalities (22) with n^2, they become:

$$\frac{1}{n}\cdot\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\le x_n\le\frac{1}{n}\cdot\sum_{k=1}^{n}f\left(\frac{k}{n}\right)+\frac{1}{2\cdot n}. \tag{23}$$

Now, passing to limit in the inequalities (23) and taking into account those presented in the Introduction, equality (19) and the fact that,

$$\lim_{n\to\infty}\frac{1}{2\cdot n}=0,$$

we obtain the equality (17). To show the equality (18), we consider the sequences:

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$$y_n=\prod_{k=1}^{n}\left[1+\frac{1}{n}\cdot f\left(\frac{k}{n}\right)\right] \quad \text{and} \quad z_n=\ln(x_n). \tag{24}$$

Then:

$$z_n=\sum_{k=1}^{n}\ln\left[1+\frac{1}{n}\cdot f\left(\frac{k}{n}\right)\right]. \tag{25}$$

Because f is integrable, it follows that f is bounded; so there is $M\in\mathbf{R}_+^*$ such that, for every $x\in[0,1]$,

$$f(x)\leq M. \tag{26}$$

Hence, according to the hypothesis, it follows that, for every $n\in\mathbf{N}$, $n>M$ and, for every $k=\overline{1,n}$,

$$0<\frac{1}{n}\cdot f\left(\frac{k}{n}\right)<1. \tag{27}$$

Then, according to the inequalities (27) and (14), for every $k=\overline{1,n}$,

$$\frac{1}{n}\cdot f\left(\frac{k}{n}\right)-\frac{1}{2\cdot n^2}\cdot f^2\left(\frac{k}{n}\right)\leq\ln\left[1+\frac{1}{n}\cdot f\left(\frac{k}{n}\right)\right]\leq\frac{1}{n}\cdot f\left(\frac{k}{n}\right). \tag{28}$$

Summing up after $k=\overline{1,n}$, member with member, the inequalities (28), we obtain:

$$\frac{1}{n}\cdot\sum_{k=1}^{n}f\left(\frac{k}{n}\right)-\frac{1}{2\cdot n^2}\cdot\sum_{k=1}^{n}f^2\left(\frac{k}{n}\right)\leq z_n\leq\frac{1}{n}\cdot\sum_{k=1}^{n}f\left(\frac{k}{n}\right). \tag{29}$$

Because f is integrable, it follows that:

$$\lim_{n\to\infty}\left[\frac{1}{n}\cdot\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\right]=\int_0^1 f(x)\cdot dx \tag{30}$$

and, according to the inequalities (26),

$$0\leq\frac{1}{2\cdot n^2}\cdot\sum_{k=1}^{n}f^2\left(\frac{k}{n}\right)\leq\frac{M^2}{2\cdot n}, \tag{31}$$

which shows that:

$$\lim_{n\to\infty}\frac{1}{2\cdot n^2}\cdot\sum_{k=1}^{n}f^2\left(\frac{k}{n}\right)=0. \tag{32}$$

Now, passing to limit in the inequalities (29) and taking into account the equality (30) and (32), we obtain that:

$$\lim_{n\to\infty}z_n=\int_0^1 f(x)\cdot dx, \tag{33}$$

which leads to equality (18).

6. Findings

In this paragraph we will present a series of sequences whose limit, after the above, can be calculated very easily. According to the Theorem, for every function that satisfies the conditions in the statement, the calculation of the limits of such sequences is reduced to the calculation of definite integral.

Therefore, as in the Theorem, considering various integrable functions on [0,1], we can conceive a

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series of sequences whose limit is given by equality (17) or equality (18). We present below a table with such results:

f : [0,1] → R	**Sequence $n^2 \cdot x_n$**	**Sequence y_n**	$\lim\limits_{n\to\infty} x_n$	$\lim\limits_{n\to\infty} y_n$
1	$\sum\limits_{k=1}^{n} \frac{1}{\ln\left(1+\frac{1}{n}\right)}$	$\prod\limits_{k=1}^{n}\left(1+\frac{1}{n}\right)$	1	e
x	$\sum\limits_{k=1}^{n} \frac{1}{\ln\left(1+\frac{1}{k}\right)}$	$\prod\limits_{k=1}^{n}\left(1+\frac{k}{n^2}\right)$	$\frac{1}{2}$	$\sqrt{e}$
x^p, $p \in \mathbf{R}_+^*$	$\sum\limits_{k=1}^{n} \frac{1}{\ln\left(1+\frac{n^{p-1}}{k^p}\right)}$	$\prod\limits_{k=1}^{n}\left(1+\frac{k^p}{n^{p+1}}\right)$	$\frac{1}{p+1}$	$e^{\frac{1}{p+1}}$
ln(1+x)	$\sum\limits_{k=1}^{n} \frac{1}{\ln\left(1+\frac{\frac{1}{n}}{\ln\left(1+\frac{k}{n}\right)}\right)}$	$\prod\limits_{k=1}^{n}\left[1+\frac{1}{n}\cdot\ln\left(1+\frac{k}{n}\right)\right]$	$2\cdot\ln2-1$	$\frac{4}{e}$
$x\cdot\ln(1+x)$	$\sum\limits_{k=1}^{n} \frac{1}{\ln\left(1+\frac{\frac{1}{k}}{\ln\left(1+\frac{k}{n}\right)}\right)}$	$\prod\limits_{k=1}^{n}\left[1+\frac{k}{n^2}\cdot\ln\left(1+\frac{k}{n}\right)\right]$	$\frac{1}{4}$	$e^{\sqrt[4]{e}}$
$\frac{\ln(1+x)}{1+x}$	$\sum\limits_{k=1}^{n} \frac{1}{\ln\left(1+\frac{\frac{n+k}{n^2}}{\ln\left(1+\frac{k}{n}\right)}\right)}$	$\prod\limits_{k=1}^{n}\left[1+\frac{1}{n+k}\cdot\ln\left(1+\frac{k}{n}\right)\right]$	$\frac{\ln^2 2}{2}$	$\sqrt{2^{\ln 2}}$
$\frac{1}{\sqrt{1+x^2}}$	$\sum\limits_{k=1}^{n} \frac{1}{\ln\left(1+\frac{\sqrt{n^2+k^2}}{n^2}\right)}$	$\prod\limits_{k=1}^{n}\left[1+\frac{1}{\sqrt{n^2+k^2}}\right]$	$\ln(\sqrt{2}+1)$	$\sqrt{2}+1$
$\frac{x}{\sqrt{1+x^2}}$	$\sum\limits_{k=1}^{n} \frac{1}{\ln\left(1+\frac{\sqrt{n^2+k^2}}{n\cdot k}\right)}$	$\prod\limits_{k=1}^{n}\left[1+\frac{k}{\sqrt{n^2+k^2}}\right]$	$\sqrt{2}-1$	$e^{\sqrt{2}-1}$
arctgx	$\sum\limits_{k=1}^{n} \frac{1}{\ln\left(1+\frac{\frac{1}{n}}{\mathrm{arctg}\frac{k}{n}}\right)}$	$\prod\limits_{k=1}^{n}\left[1+\frac{1}{n}\cdot\mathrm{arctg}\frac{k}{n}\right]$	$\frac{\pi}{4}-\frac{\ln 2}{2}$	$\frac{\sqrt[4]{4\cdot e^{\pi}}}{2}$

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$x\cdot \text{arctg}x$	$\sum_{k=1}^{n}\frac{1}{\ln\left(1+\frac{\frac{1}{k}}{\text{arctg}\frac{k}{n}}\right)}$	$\prod_{k=1}^{n}\left[1+\frac{k}{n^2}\cdot\text{arctg}\frac{k}{n}\right]$	$\frac{\pi}{4}-\frac{1}{2}$	$\sqrt[4]{e^{\pi-2}}$
$x\cdot e^x$	$\sum_{k=1}^{n}\frac{1}{\ln\left(1+\frac{\frac{1}{k}}{e^{\frac{k}{n}}}\right)}$	$\prod_{k=1}^{n}\left[1+\frac{k}{n^2}\cdot e^{\frac{k}{n}}\right]$	1	e
$\sqrt{1+x^2}$	$\sum_{k=1}^{n}\frac{1}{\ln\left(1+\frac{1}{\sqrt{n^2+k^2}}\right)}$	$\prod_{k=1}^{n}\left[1+\frac{\sqrt{n^2+k^2}}{n^2}\right]$	$\frac{\sqrt{2}+\ln(\sqrt{2}+1)}{2}$	$\sqrt{e^{\sqrt{2}}\cdot(\sqrt{2}+1)}$
$x\cdot\sqrt{1+x^2}$	$\sum_{k=1}^{n}\frac{1}{\ln\left(1+\frac{n}{k\cdot\sqrt{n^2+k^2}}\right)}$	$\prod_{k=1}^{n}\left[1+\frac{k\cdot\sqrt{n^2+k^2}}{n^3}\right]$	$\frac{2\cdot\sqrt{2}-1}{3}$	$\sqrt[3]{e^{\sqrt{2}-1}}$
$\sqrt{1-x^2}$	$\sum_{k=1}^{n}\frac{1}{\ln\left(1+\frac{1}{\sqrt{n^2-k^2}}\right)}$	$\prod_{k=1}^{n}\left[1+\frac{\sqrt{n^2-k^2}}{n^2}\right]$	$\frac{\pi}{4}$	$\sqrt[4]{e^{\pi}}$
$x\cdot\sqrt{1-x^2}$	$\sum_{k=1}^{n}\frac{1}{\ln\left(1+\frac{n}{k\cdot\sqrt{n^2-k^2}}\right)}$	$\prod_{k=1}^{n}\left[1+\frac{k\cdot\sqrt{n^2-k^2}}{n^3}\right]$	$\frac{1}{3}$	$\sqrt[3]{e}$

Let's leave the calculations from the table above on the reader's account, and we calculate the limits of the following four sequences:

$$a_n=\frac{1}{n^2}\cdot\sum_{k=1}^{n}\frac{1}{\ln\left(1+\frac{n^2+k^2}{n^2\cdot k}\right)},\qquad b_n=\frac{1}{n^2}\cdot\sum_{k=1}^{n}\frac{1}{\ln\left(1+\frac{(k+n)\cdot\sqrt{k^2+2\cdot n\cdot k+2\cdot n^2}}{n^3}\right)},$$

$$c_n=\prod_{k=1}^{n}\left[1+\frac{k\cdot n}{n^2+k^2}\right] \quad\text{and}\quad d_n=\prod_{k=1}^{n}\left[1+\frac{n}{(k+n)\cdot\sqrt{k^2+2\cdot n\cdot k+2\cdot n^2}}\right].$$

Of course that we will apply the above Theorem. In this regard, we observe that:

$$a_n=\frac{1}{n^2}\cdot\sum_{k=1}^{n}\frac{1}{\ln\left(1+\frac{\frac{1}{n}}{\frac{k}{n}\cdot\left(1+\frac{k^2}{n^2}\right)^{-1}}\right)},\qquad b_n=\frac{1}{n^2}\cdot\sum_{k=1}^{n}\frac{1}{\ln\left(1+\frac{\frac{1}{n}}{\left[\left(1+\frac{k}{n}\right)\cdot\sqrt{\left(\frac{k}{n}+1\right)^2+1}\right]^{-1}}\right)},$$

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$$c_n=\prod_{k=1}^{n}\left[1+\frac{1}{n}\cdot\frac{\frac{k}{n}}{1+\frac{k^2}{n^2}}\right] \quad \text{and} \quad d_n=\prod_{k=1}^{n}\left[1+\frac{\frac{1}{n}}{\left[\left(\frac{k}{n}+1\right)\cdot\sqrt{\left(\frac{k}{n}+1\right)^2+1}\right]^{-1}}\right].$$

So, we consider the functions:

$p : [0,1] \to \mathbf{R}$, and $q : [0,1] \to \mathbf{R}$,

where, for every $x\in[0,1]$,

$$p(x)=\frac{x}{1+x^2} \quad \text{and} \quad q(x)=\frac{1}{(x+1)\cdot\sqrt{(x+1)^2+1}}.$$

Since:

$$\int_0^1 m(x)\cdot dx = \ln\sqrt{2} \quad \text{and} \quad \int_0^1 n(x)\cdot dx = \ln\frac{(\sqrt{5}-1)\cdot(\sqrt{2}+1)}{2},$$

according to the Theorem, we have the equalities:

$$\lim_{n\to\infty} a_n=\ln\sqrt{2}, \qquad \lim_{n\to\infty} b_n=\ln\frac{(\sqrt{5}-1)\cdot(\sqrt{2}+1)}{2},$$

$$\lim_{n\to\infty} c_n=\sqrt{2} \quad \text{and} \quad \lim_{n\to\infty} d_n=\frac{(\sqrt{5}-1)\cdot(\sqrt{2}+1)}{2}.$$

We propose to the reader to solve the next exercise:

***Exercise*:** Calculate the limits of the following sequences:

$$\alpha_n=\frac{1}{n^2}\cdot\sum_{k=1}^{n}\frac{1}{\ln\left(1+\frac{n^2+k^2}{n^3\cdot(\ln(n+k)-\ln n)}\right)}, \quad \beta_n=\frac{1}{n^2}\cdot\sum_{k=1}^{n}\frac{1}{\ln\left(1+\frac{(k+n)\cdot(k+2\cdot n)\cdot\Lambda\cdot(k+p\cdot n+n)}{n^{p+2}}\right)}$$

$$\chi_n=\prod_{k=1}^{n}\left[1+\frac{n\cdot(\ln(n+k)-\ln n)}{n^2+k^2}\right] \quad \text{and} \quad \delta_n=\prod_{k=1}^{n}\left[1+\frac{n^p}{(k+n)\cdot(k+2\cdot n)\cdot\Lambda\cdot(k+n\cdot p+n)}\right], \text{ with } p\in\mathbf{N}^*.$$

***Hint*:** Reasoning as in the examples above, for the sequences α_n and χ_n using the function:

$$q : [0,1] \to \mathbf{R}, \qquad q(x)=\frac{\ln(1+x)}{1+x^2},$$

you will obtain, using the Theorem, that the sequence α_n has the limit equal to $\frac{\pi}{8}\cdot\ln 2$, and the sequence χ_n has the limit equal to $\sqrt[8]{2^\pi}$. For the sequences β_n and δ_n using the function:

$$r : [0,1] \to \mathbf{R}, \qquad r(x)=\frac{1}{(x+1)\cdot(x+2)\cdot\Lambda\cdot(x+p)\cdot(x+p+1)},$$

you will obtain, using the Theorem and the relation (29) from Vălcan (2016), that the required limits are $\ln\sqrt[p!]{\prod_{k=1}^{n}\left(\frac{k+1}{k+2}\right)^{(-1)^k\cdot C_p^k}}$, respectively $\sqrt[p!]{\prod_{k=1}^{n}\left(\frac{k+1}{k+2}\right)^{(-1)^k\cdot C_p^k}}$.

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7. Conclusion

Of course, the main results presented in this paper, through their applications to the calculation of some categories of limits of sequences, will lead to the development of methodological competences to solve many problems of Mathematics. Moreover, the attentive reader interested in these issues will notice that the inequalities (14) derive from the Taylor series development of the function ln(1+x), according to which (see Taylor's Theorem):

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n-1}\cdot\frac{x^n}{n}+\cdots$$

and considering other functions than those in the table above, will get other limitis of sequences, thus contributing to the development of the creative character of mathematical education (http://mathworld.wolfram.com/TaylorsTheorem.html).

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