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STRUCTURES OF FIELDS OF RATIONAL NUMBERS, ISOMORPHIC BETWEEN THEM

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Abstract

As it is known, (I mentioned this) often solving a problem / exercise in a certain algebraic structure is quite difficult. That is why it is sometimes necessary to transfer the respective problem / exercise to an isomorphic structure with the given one and where it can be solved / studied more easily. But the problem of determining isomorphic algebraic structures at once is quite difficult for both pupils and students and teachers. In this paper we propose to build other structures of isomorphic (commutative) field with the field of rational numbers Q, on different subsets of the set Q, structures different from those known / presented so far. To begin with, we will see that if a and b are any two rational numbers, then on the set of rational numbers at most equal to a, so on the set of Q-∞,a, and on the set of rational numbers at least equal to b, so on the set Qb,+∞, we will be able to define such a structure. Moreover, all new structures of fields defined here will be isomorphic to each other. It will result, then, that there is a double infinity of structures of (commutative) fields of rational numbers, all isomorphic to each other, but also to the field (Q,+,·).

Keywords: Didactics of Mathematics, field, isomorphism, mathematics, rational number

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1. Introduction

To begin with, we will try to answer the following question:

What influence should logic have on the learning of Mathematics?

This question shapes the sector of a fundamental problem of education through Mathematics, because not every notional mathematical content can be learned at once and, consequently, the content of the instruction must be ordered in a certain logical sequence.

When the content of the training is logically ordered, it is arranged according to a hierarchy of principles and concepts that are supposed to be part of the discipline itself. In general, the mathematical knowledge taught in school is organized in a number of mathematical subdisciplines:

i. Arithmetic,
ii. Algebra,
iii. Geometry, ...,

Each being both a field of knowledge and a way of knowing. Each mathematical subdiscipline includes a way of thinking or investigating a world, a way that has proven its functionality over time. As the knowledge of Mathematics progresses, new sub-domains open up in front of the investigation:

i. Trigonometry,
ii. Analytical geometry,
iii. Mathematical analysis,

and where existing methods cannot be extended, new disciplines are defined:

i. Mathematics in art,
ii. Astronomy.

Sometimes established disciplines, such as Projective Geometry, are abandoned. It would therefore follow that the mathematical subdisciplines, at least some of them, do not represent permanent ways of thinking, although it is difficult to imagine how we could dispense with the study of Algebra or Geometry, for example. They are forms of investigation that have proven to be the most effective over time, although they are likely to be reviewed at any time (Vălcan, 2022).

We can say from the perspective of cultural experience that mathematical subdisciplines have proven to be the most efficient way to acquire and organize mathematical knowledge. We can formulate from this experience the premise that those who will learn Mathematics in this framework will acquire their knowledge more efficiently, if they analyze the methods by which this knowledge was discovered and the structures according to which it was organized.

Therefore, in order to be mastered and used, mathematical knowledge must be organized in the lesson, largely similar to the way they are organized, in Mathematical Science. They remain the essential determinant, even if we cannot say that they represent the ideal order of school learning. Other factors must be taken into account in school learning, how are the psychological ones,

i. a degree of maturity of the student,
ii. skills
iii. his motivation,

factors that act in the manner of conditioning, by reference to the determining character of the mathematical subdisciplines (Astolfi & Develey, 1989).

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The term school learning, in general, refers to the process of mnemonic acquisition, active assimilation of information, formation of intellectual operations, motor skills and attitudes. As you can see, we define the school learning process through its product:

i. information,
ii. intellectual operations,
iii. motor skills,
iv. attitudes,

so depending on what is learned, the content – see (Ausubel, 1968). There is, however, a relative autonomy of the learning process from the product, which results in the existence of legitimacy common to the process as a whole. A study of facts and principles has emerged from the study of learning processes, which remain valid in a wide variety of situations. These make up the theory of learning (or epistemology) whose data and conclusions are taken from and incorporated into Didactics, in particular, in Didactics of Mathematics.

It would be wrong to equate learning with memorization. As some experts note, learning Mathematics is not limited to simply "*storing*" the information transmitted by the teacher, storing this information in the "*memory - storage*" of the student or updating it when checking the data purchased. Not infrequently, in school practice, there is such an equivalence between learning Mathematics and various memory functions (recognition, preservation or reproduction of mathematical knowledge). The acquisition of mathematical knowledge, the formation of skills to solve exercises and problems, are not just memory tasks; attention intervenes here - as a first condition of learning - then perception (in the form of observation) and thinking - with its operations of analysis, synthesis and generalization - followed by fixation in memory – see (Vălcan, 1997).

1. Problem Statement

Starting from those presented in the previous paragraph, but within a more general context, that of training and developing the competencies of pupils and students and teachers of solving Mathematics problems, we propose in this paper the construction of new body structures. rational numbers, isomorphic to each other, but also isomorphic to the field $(Q,+,\cdot)$.

But, as it is known, and as I mentioned above, often solving a problem / exercise in a certain algebraic structure is quite difficult. That is why it is sometimes necessary to transfer the respective problem / exercise to an isomorphic structure with the given one and where it can be solved / studied more easily. But the problem of determining isomorphic algebraic structures at once is quite difficult for both pupils and students and teachers.

In this paper we propose to build other structures of isomorphic (commutative) field with the field of rational numbers Q, on different subsets of the set Q, structures different from those known / presented so far. To begin with, we will see that if a and b are any two rational numbers, then on the set of rational numbers at most equal to a, so on the set of $Q_{-\infty,a}$, and on the set of rational numbers at least equal to b, so on the set $Q_{b,+\infty}$, we will be able to define such a structure. Moreover, all new structures of fields defined here will be isomorphic to each other. It will result, then, that there is a double infinity of structures of (commutative) fields of rational numbers, all isomorphic to each other, but also to the field $(Q,+,\cdot)$.

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Of course, we will use the ideas from the other papers that focused on similar aspects: first of all, the paper (Vălcan, 2017), which was the basis for the elaboration of other similar papers. We refer here to the work (Vălcan, 2018), where we showed that there is an isomorphic field of functions with the field of real numbers ($\mathbf{R}$,+,·), then the papers (Vălcan, 2020) and (Vălcan, 2021), where we showed that there are a series of rings of integer numbers, isomorphic between them, but also isomorphic with the ring of integer numbers ($\mathbf{Z}$,+,·).

2. Research Questions

In our research we will try to find answers to the following questions:

i. There are others structures of field defined on sets of rational numbers, apart from the known ones, and which are isomorphic to the field of rationals, (Q,+,·)?
ii. How can these structures be identified?

2.1. Regarding the first question

We are thinking here of sets of rational numbers, unbounded inferior, but bordered superiorly, or vice versa.

2.2. Regarding the second question

We refer here to the ways of determining both these structures and the isomorphisms between them.

3. Purpose of the Study

Therefore, we answered the two questions in Paragraph 3. Thus, for any numbers a, b∈Q there are two pairs of laws of internal composition on the sets $Q_{-\infty,a}$ and $Q_{b,+\infty}$, let's say „♣" and „♦", respectively „♥" and „♠", so that ($Q_{-\infty,a}$,♣,♦) and ($Q_{b,+\infty}$,♥,♠,) become commutative fields isomorphic to the field (Q,+,·).

Concretely, on the set of rationals at most equal to 3, which we denote with $\mathbf{Q}_{-\infty,3}$ and on the set of integers at least equal to 5, which we denote with $\mathbf{Q}_{5,+\infty}$, we can define two pairs of laws of internal composition so that let's say „♣" and „♦", respectively „♥" and „♠", so that ($\mathbf{Q}_{-\infty,3}$,♣,♦) and ($\mathbf{Q}_{5,+\infty}$,♥,♠) become commutative rings isomorphic to the field ($\mathbf{Q}$,+,·).

4. Research Methods

Let be a, b∈$\mathbf{Q}$. We note with:

$\mathbf{Q}_{-\infty,a}=\{x\in\mathbf{Q} \mid x<a\}$ and $\mathbf{Q}_{b,+\infty}=\{x\in\mathbf{Q} \mid x>b\}$.

Then the functions:

$f_a : \mathbf{Q} \to \mathbf{Q}_{-\infty,a}$ and $g_b : \mathbf{Q} \to \mathbf{Q}_{b,+\infty}$,

defined by:

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$$f_a(x)=\begin{cases} a+\dfrac{1}{x}, & \text{if } x\in(-\infty,-1] \\ a+x-1, & \text{if } x\in(-1,0) \\ a-x-2, & \text{if } x\in[0,+\infty) \end{cases} \quad \text{and} \quad g_b(x)=\begin{cases} b-\dfrac{1}{x}, & \text{if } x\in(-\infty,-1] \\ b-x+1, & \text{if } x\in(-1,0) \\ b+x+2, & \text{if } x\in[0,+\infty) \end{cases}$$

are bijections, and their inverses are functions:

$$f_a^{-1}:\mathbf{Q}_{-\infty,a}\to\mathbf{Q} \quad \text{and} \quad g_b^{-1}:\mathbf{Q}_{b,+\infty}\to\mathbf{Q},$$

defined by:

$$f_a^{-1}(x)=\begin{cases} a-x-2, & \text{if } x\in(-\infty,a-2] \\ x-a+1, & \text{if } x\in(a-2,a-1) \\ \dfrac{1}{x-a}, & \text{if } x\in[a-1,a) \end{cases} \quad \text{and} \quad g_b^{-1}(x)=\begin{cases} \dfrac{1}{b-x}, & \text{if } x\in(b,b+1] \\ b-x+1, & \text{if } x\in(b+1,b+2) \\ x-b-2, & \text{if } x\in[b+2,+\infty) \end{cases},$$

which, according to Vălcan, (2019) shows that:

$$\mathbf{Q}\sim\mathbf{Q}_{-\infty,a} \quad \text{and} \quad \mathbf{Q}\sim\mathbf{Q}_{b,+\infty},$$

whence it follows that:

$$\mathbf{Q}_{-\infty,a}\sim\mathbf{Q}_{b,+\infty};$$

the bijection that accomplishes this is:

$$h_{a,b}=g_b\circ f_a^{-1}:\mathbf{Q}_{-\infty,a}\to\mathbf{Q}_{b,+\infty};$$

where, for every $x\in\mathbf{Q}_{-\infty,a}$,

$$h_{a,b}(x)=(g_b\circ f_a^{-1})(x)=g_b(f_a^{-1}(x))=\begin{cases} b-\dfrac{1}{f_a^{-1}(x)}, & \text{if } f_a^{-1}(x)\in(-\infty,-1] \\ b-f_a^{-1}(x)+1, & \text{if } f_a^{-1}(x)\in(-1,0) \\ b+f_a^{-1}(x)+2, & \text{if } f_a^{-1}(x)\in[0,+\infty) \end{cases}$$

$$=\begin{cases} b-x+a, & \text{if } x\in[a-1,a) \\ b-x+a-1+1, & \text{if } x\in(a-2,a-1) \\ b+a-x-2+2, & \text{if } x\in(-\infty,a-2] \end{cases}=a+b-x=b+(a-x),$$

and

$$h_{a,b}^{-1}:f_a\circ g_b^{-1}:\mathbf{Q}_{b,+\infty}\to\mathbf{Q}_{-\infty,a}; \quad \text{where, for every } x\in\mathbf{Q}_{-\infty,a}, \quad h_{a,b}^{-1}(x)=a-(x-b).$$

It follows that the following diagram (A) in Figure 1, is commutative:

$$\begin{tikzcd} \mathbf{Q} \arrow[r, "f_a"] \arrow[d, "g_b"'] & \mathbf{Q}_{-\infty,a} \arrow[dl, "h_{a,b}=g_b\circ f_a^{-1}."] \\ \mathbf{Q}_{b,+\infty} & \end{tikzcd}$$

Figure 1. Diagram (A)

The first fundamental result of this paragraph is:

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Theorem 5.1: *For every number $a \in \mathbf{Q}$, there are two laws of internal composition, let's say „♣” and „♦”, on the set $\mathbf{Q}_{-\infty,a}$, such that ($\mathbf{Q}_{-\infty,a}$, ♣, ♦) to become is a (commutative) field isomorphic to the field ($\mathbf{Q}$,+,·).*

Proof: We transfer the field structure from **Q** to $\mathbf{Q}_{-\infty,\mathbf{a}}$, using the functions:

$f_a : \mathbf{Q} \to \mathbf{Q}_{-\infty,a}$, where, for every $x \in \mathbf{Q}$,
$$f_a(x)=\begin{cases} a+\dfrac{1}{x} & \text{, if } x \in (-\infty,-1] \\ a+x-1, & \text{if } x \in (-1,0) \\ a-x-2, & \text{if } x \in [0,+\infty) \end{cases}$$

and

$f_a^{-1} : \mathbf{Q}_{-\infty,a} \to \mathbf{Q}$, is defined by:
$$f_a^{-1}(x)=\begin{cases} a-x-2, & \text{if } x \in (-\infty, a-2] \\ x-a+1, & \text{if } x \in (a-2, a-1) \\ \dfrac{1}{x-a} & \text{, if } x \in [a-1, a) \end{cases}.$$

So, according to Vălcan (2017), we obtain the two laws of composition „♣” and „♦” on the set of integers $\mathbf{Q}_{-\infty,a}$. Let be $x, y \in \mathbf{Q}_{-\infty,a}$. For defining the law „♣”, we distinguish the following cases:

<u>Case 1</u>: $x, y \in (-\infty, a-2]$. Then:

$$x♣y=f_a(f_a^{-1}(x)+f_a^{-1}(y))=f_a((a-x-2)+(a-y-2))=f_a(2\cdot a-x-y-4)=a-(2\cdot a-x-y-4)-2$$
$$=a-(a-x)-(a-y)+2.$$

<u>Case 2</u>: $x \in (-\infty, a-2]$ and $y \in (a-2, a-1)$. Then:

$$x♣y=f_a(f_a^{-1}(x)+f_a^{-1}(y))=f_a((a-x-2)+(y-a+1))=f_a(-x+y-1)$$
$$=\begin{cases} a+(a-x)-(a-y)-2, & \text{if } y < x+1 \\ a-(a-x)+(a-y)-1, & \text{if } y \geq x+1 \end{cases}.$$

<u>Case 3</u>: $x \in (-\infty, a-2]$ and $y \in [a-1, a)$. Then:

$$x♣y=f_a(f_a^{-1}(x)+f_a^{-1}(y))=f_a\left((a-x-2)-\frac{1}{a-y}\right)$$
$$=\begin{cases} a+\dfrac{a-y}{[(a-x)-2]\cdot(a-y)-1} & \text{, if } [(a-x)-1]\cdot(a-y) \leq -1 \\ a+(a-x)-\dfrac{1}{a-y}-3 & \text{, if } \begin{cases} [(a-x)-1]\cdot(a-y) > 1 \\ [(a-x)-2]\cdot(a-y) < 1 \end{cases} \\ a-(a-x)+\dfrac{1}{a-y} & \text{, if } [(a-x)-2]\cdot(a-y) \geq 1 \end{cases}.$$

<u>Case 4</u>: $x, y \in (a-2, a-1)$. Then:

$$x♣y=f_a(f_a^{-1}(x)+f_a^{-1}(y))=f_a((x-a+1)+(y-a+1))=f_a(x+y-2\cdot a+2)$$
$$=\begin{cases} a-\dfrac{1}{(a-x)+(a-y)-2} & \text{, if } 3 \leq (a-x)+(a-y) < 4 \\ a-(a-x)-(a-y)+1 & \text{, if } 2 < (a-x)+(a-y) < 3 \end{cases}.$$

<u>Case 5</u>: $x \in (a-2, a-1)$ and $y \in [a-1, a)$. Then:

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$$x♣y=f_a(f_a^{-1}(x)+f_a^{-1}(y))=f_a\left((x-a+1)-\frac{1}{a-y}\right)=a+(a-x)+\frac{1}{a-y}-3.$$

Case 6: x, $y\in[a-1,a)$. Then:

$$x♣y=f_a(f_a^{-1}(x)+f_a^{-1}(y))=f_a\left(-\frac{1}{a-x}-\frac{1}{a-y}\right)=a-\frac{(a-x)\cdot(a-y)}{(a-x)+(a-y)}.$$

Therefore, for every x, $y\in\mathbf{Q}_{-\infty,a}$,

$x♣y=a-(a-x)-(a-y)+2$, if x, $y\in(-\infty,a-2]$;

$$x♣y=\begin{cases}a+(a-x)-(a-y)-2, \text{ if } y<x+1\\ a-(a-x)+(a-y)-1, \text{ if } y\geq x+1\end{cases} \text{ and } x\in(-\infty,a-2],\ y\in(a-2,a-1);$$

$$x♣y=\begin{cases}a+\dfrac{a-y}{[(a-x)-2]\cdot(a-y)-1}, \text{ if } [(a-x)-1]\cdot(a-y)\leq 1\\ a+[(a-x)-3]-\dfrac{1}{a-y}, \text{ if } \begin{cases}[(a-x)-1]\cdot(a-y)>1\\ [(a-x)-2]\cdot(a-y)<1\end{cases}\\ a-(a-x)+\dfrac{1}{a-y}, \text{ if } [(a-x)-2]\cdot(a-y)\geq 1\end{cases} \text{ and } x\in(-\infty,a-2],\ y\in[a-1,a);$$

$$x♣y=\begin{cases}a-\dfrac{1}{(a-x)+(a-y)-2}, \text{ if } 3\leq(a-x)+(a-y)<4\\ a-(a-x)-(a-y)+1, \text{ if } 2<(a-x)+(a-y)<3\end{cases} \text{ and } x,\ y\in(a-2,a-1);$$

$x♣y=a+(a-x)+\dfrac{1}{a-y}-3$, if $x\in(a-2,a-1)$ and $y\in[a-1,a)$;

$x♣y=a-\dfrac{(a-x)\cdot(a-y)}{(a-x)+(a-y)}$ and x, $y\in[a-1,a)$.

Now, for defining the law „♦", we distinguish the following cases:

Case 1: x, $y\in(-\infty,a-2]$. Then:

$$x♦y=f_a(f_a^{-1}(x)\cdot f_a^{-1}(y))=f_a((a-x-2)\cdot(a-y-2))=a-(a-x-2)\cdot(a-y-2)-2$$
$$=a-(a-x)\cdot(a-y)+2\cdot[(a-x)+(a-y)]-6.$$

Case 2: $x\in(-\infty,a-2]$ and $y\in(a-2,a-1)$. Then:

- if x=a-2, then:

  $x♦y=f_a(f_a^{-1}(x)\cdot f_a^{-1}(y))=f_a(0\cdot(y-a+1))=f_a(0)=a-2$,
- if x<a-2, then:

$$x♦y=\begin{cases}a+(a-x-2)\cdot(y-a+1)-1, \text{ if } [(a-x)-2]\cdot[(a-y)-1]<1\\ a-\dfrac{1}{(a-x-2)\cdot(a-y-1)}, \text{ if } [(a-x)-2]\cdot[(a-y)-1]\geq 1\end{cases}$$

$$=\begin{cases}a-(a-x)\cdot(a-y)+(a-x)+2\cdot(a-y)-3, \text{ if } [(a-x)-2]\cdot[(a-y)-1]<1\\ a-\dfrac{1}{[(a-x)-2]\cdot[(a-y)-1]}, \text{ if } [(a-x)-2]\cdot[(a-y)-1]\geq 1\end{cases}$$

Case 3: $x\in(-\infty,a-2]$ and $y\in[a-1,a)$. Then:

- if x=a-2, then:

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$$x\blacklozenge y=f_a(f_a^{-1}(x)\cdot f_a^{-1}(y))=f_a\left(0\cdot\frac{-1}{a-y}\right)=f_a(0)=a-2,$$

- if x<a-2, then:

$$x\blacklozenge y=f_a(f_a^{-1}(x)\cdot f_a^{-1}(y))=f_a\left((a-x-2)\cdot\frac{-1}{a-y}\right)=\begin{cases}a-\dfrac{(a-x)-2}{a-y}-1, \text{if } y<x+2\\ a-\dfrac{(a-y)}{(a-x)-2}, \text{if } y\geq x+2\end{cases}.$$

<u>Case 4</u>: x, y∈(a-2,a-1). Then:

$$x\blacklozenge y=f_a(f_a^{-1}(x)\cdot f_a^{-1}(y))=f_a((x-a+1)\cdot(y-a+1))=a+(x-a+1)\cdot(y-a+1)-1$$

$$=a+[(a-x)-1]\cdot[(a-y)-1]-1=a+(a-x)\cdot(a-y)-(a-x)-(a-y).$$

<u>Case 5</u>: x∈(a-2,a-1) and y∈[a-1,a). Then:

$$x\blacklozenge y=f_a(f_a^{-1}(x)\cdot f_a^{-1}(y))=f_a\left((x-a+1)\cdot\frac{-1}{a-y}\right)=f_a\left(\frac{(a-x)-1}{a-y}\right)=a-\frac{(a-x)-1}{a-y}-2.$$

<u>Case 6</u>: x, y∈[a-1,a). Then:

$$x\blacklozenge y=f_a(f_a^{-1}(x)\cdot f_a^{-1}(y))=f_a\left(\frac{1}{a-x}\cdot\frac{1}{a-y}\right)=a-\frac{1}{(a-x)\cdot(a-y)}-2.$$

Therefore, for every x, y∈$\mathbf{Q}_{-\infty,a}$,

$x\blacklozenge y=a-(a-x)\cdot(a-y)+2\cdot[(a-x)+(a-y)]-6$, if x, y∈(-∞,a-2];

- if x=a-2 and y∈(a-2,a-1), then:

  $x\blacklozenge y=(a-2)\blacklozenge y=a-2$;

- x∈(-∞,a-2) and y∈(a-2,a-1), then:

$$x\blacklozenge y=\begin{cases}a-(a-x)\cdot(a-y)+(a-x)+2\cdot(a-y)-3, \text{if } [(a-x)-2]\cdot[(a-y)-1]<1\\ a-\dfrac{1}{[(a-x)-2]\cdot[(a-y)-1]}, \text{if } [(a-x)-2]\cdot[(a-y)-1]\geq 1\end{cases};$$

- if x=a-2 and y∈[a-1,a), then:

  $x\blacklozenge y=(a-2)\blacklozenge y=a-2$;

- if x∈(-∞,a-2) and y∈[a-1,a), then:

$$x\blacklozenge y=\begin{cases}a-\dfrac{(a-x)-2}{a-y}-1, \text{if } y<x+2\\ a-\dfrac{(a-y)}{(a-x)-2}, \text{if } y\geq x+2\end{cases};$$

$x\blacklozenge y=a+(a-x)\cdot(a-y)-(a-x)-(a-y)$, if x, y∈(a-2,a-1);

$x\blacklozenge y=a-\dfrac{(a-x)-1}{a-y}-2$, if x∈(a-2,a-1) and y∈[a-1,a);

$x\blacklozenge y=a-\dfrac{1}{(a-x)\cdot(a-y)}-2$, if x, y∈[a-1,a).

On the other hand,

$e_{Q_{-\infty,a}}=f_a(e_{\mathbf{Q}})=f_a(0)=a-2$

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and

$$-x_{Q_{-\infty,a}}=f_a(-f_a^{-1}(x))=\begin{cases} a-\dfrac{1}{(a-x)-2}, & \text{if } x\in(-\infty,a-3] \\ x+1 & ,\text{if } x\in(a-3,a-2) \\ a-2 & ,\text{if } x=a-2 \\ x-1 & ,\text{if } x\in(a-2,a-1) \\ a-\dfrac{1}{a-x}-2 & ,\text{if } x\in[a-1,a) \end{cases},$$

and:

$$1_{Q_{-\infty,a}}=f_a(1_{\mathbf{Q}})=a-3$$

and, for every $x\in\mathbf{Q}_{-\infty,a}\setminus\{a-2\}$,

$$x^{-1}_{Q_{-\infty,a}}=f_a\left(\frac{1}{f_a^{-1}(x)}\right)=\begin{cases} a-\dfrac{1}{(a-x)-2}-2, & \text{if } x\in(-\infty,a-2) \\ a-\dfrac{1}{(a-x)-1} & ,\text{if } x\in(a-2,a-1) \\ a-1 & ,\text{if } x=a-1 \\ x-1 & ,\text{if } x\in(a-1,a) \end{cases}.$$

Therefore, according to Vălcan (2017), ($\mathbf{Q}_{-\infty,a}$,♣,♦) is a (commutative) field isomorphic to the field of rational numbers, ($\mathbf{Q}$,+,·).

Now let's show that, indeed, the function:

$f_a:\mathbf{Q}\to\mathbf{Q}_{-\infty,a}$, defined by: $$f_a(x)=\begin{cases} a+\dfrac{1}{x} & ,\text{if } x\in(-\infty,-1] \\ a+x-1, & \text{if } x\in(-1,0) \\ a-x-2, & \text{if } x\in[0,+\infty) \end{cases},$$

is an isomorphism between the two fields. For this, we first notice that, for every x, y∈$\mathbf{Q}$,

$$f_a(x+y)=\begin{cases} a+\dfrac{1}{x+y} & ,\text{if } x+y\in(-\infty,-1] \\ a+x+y-1, & \text{if } x+y\in(-1,0) \\ a-x-y-2, & \text{if } x+y\in[0,+\infty) \end{cases}$$ and $$f_a(x\cdot y)=\begin{cases} a+\dfrac{1}{x\cdot y} & ,\text{if } x\cdot y\in(-\infty,-1] \\ a+x\cdot y-1, & \text{if } x\cdot y\in(-1,0) \\ a-x\cdot y-2, & \text{if } x\cdot y\in[0,+\infty) \end{cases}.$$

To determine the expressions $f_a(x)$♣$f_a(y)$, respectively $f_a(x)$♦$f_a(y)$, we distinguish the following cases:

Case 1: x, y∈(-∞,-1]. Then:

$f_a(x)=a+\dfrac{1}{x}\in[a-1,a)$ and $f_a(y)=a+\dfrac{1}{y}\in[a-1,a)$.

It follows that:

$f_a(x)$♣$f_a(y)=a+\dfrac{1}{x+y}$ and $f_a(x)$♦$f_a(y)=a-x\cdot y-2$.

Case 2: x∈(-∞,-1] and y∈(-1,0). Then:

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$f_a(x)=a+\frac{1}{x}\in[a-1,a)$ and $f_a(y)=a+y-1\in(a-2,a-1)$.

It follows that:

$f_a(x)♣f_a(y)=a-x-y-2$ and $f_a(x)♦f_a(y)=a-x\cdot y-2$.

Case 3: $x\in(-\infty,-1]$ and $y\in[0,+\infty)$. Then:

$f_a(x)=a+\frac{1}{x}\in[a-1,a)$ and $f_a(y)=a-y-2\in(-\infty,a-2]$.

It follows that:

$$f_a(x)♣f_a(y)=\begin{cases} a+\frac{1}{x+y} & \text{, if } \frac{y+1}{x}\geq 1 \\ a+x+y-1 & \text{, if } x+y\in(-1,0) \\ a-x-y-2, & \text{if } x+y\geq 0 \end{cases}=\begin{cases} a+x+y-1 & \text{, if } x+y\in(-1,0) \\ a-x-y-2, & \text{if } x+y\geq 0 \end{cases},$$

because the first option is not possible, and

$$f_a(x)♦f_a(y)=\begin{cases} a+x\cdot y-1, & \text{if } \frac{x\cdot y+1}{x}<0 \\ a+\frac{1}{x\cdot y} & \text{, if } x\frac{x\cdot y+1}{x}\geq 0 \end{cases}=\begin{cases} a+x\cdot y-1, & \text{if } x\cdot y>-1 \\ a+\frac{1}{x\cdot y} & \text{, if } x\cdot y\leq -1 \end{cases}.$$

Case 4: $x, y\in(-1,0)$. Then:

$f_a(x)=a+x-1\in(a-2,a-1)$ and $f_a(y)=a+y-1\in(a-2,a-1)$.

It follows that:

$$f_a(x)♣f_a(y)=\begin{cases} a+\frac{1}{x+y} & \text{, if } -2\leq x+y\leq -1 \\ a+x+y-1, & \text{if } x+y\in(-1,0) \end{cases}$$ and $f_a(x)♦f_a(y)=a+x\cdot y-1$.

Case 5: $x\in(-1,0)$ and $y\in[0,+\infty)$. Then:

$f_a(x)=a+x-1\in(a-2,a-1)$ and $f_a(y)=a-y-2\in(-\infty,a-2]$.

It follows that:

$$f_a(x)♣f_a(y)=\begin{cases} a+x+y-1, & \text{if } x+y<0 \\ a+x+y-2, & \text{if } x+y>0 \end{cases}$$ and $$f_a(x)♦f_a(y)=\begin{cases} a+x\cdot y-1, & \text{if } x\cdot y>-1 \\ a+\frac{1}{x\cdot y} & \text{, if } x\cdot y\leq -1 \end{cases}.$$

Case 6: $x, y\in[0,+\infty)$. Then:

$f_a(x)=a-x-2\in(-\infty,a-2]$ and $f_a(y)=a-y-2\in(-\infty,a-2]$.

It follows that:

$f_a(x)♣f_a(y)=a-x-y-2$ and $f_a(x)♦f_a(y)=a-x\cdot y-2$.

It follows that for every $x, y\in\mathbf{Q}$:

$f_a(x+y)=f_a(x)♣f_a(y)$ and $f_a(x\cdot y)=f_a(x)♦f_a(y)$.

Therefore, according to Vălcan (2017), ($\mathbf{Q}_{-\infty,a}$,♣,♦) is a (commutative) field isomorphic to the field of rational nambers, ($\mathbf{Q}$,+,·).

Remark 5.2: *For a=0, from Theorem 5.1, obtain the field structure on the set* $\mathbf{Q}_{-\infty,0}$, *of negative rational numbers, transferred from the filed* ($\mathbf{Q}$,+,·) *by function*:

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$f_0 : \mathbf{Q} \to \mathbf{Q}_{-\infty,0}$, defined by: $f_0(x)=\begin{cases} \frac{1}{x}, & \text{if } x \in (-\infty,-1] \\ x-1, & \text{if } x \in (-1,0) \\ -x-2, & \text{if } x \in [0,+\infty) \end{cases}$

and for which the inverse is:

$f_0^{-1} : \mathbf{Q}_{-\infty,0} \to \mathbf{Q}$, defined by: $f_0^{-1}(x)=\begin{cases} -x-2, & \text{if } x \in (-\infty,-2] \\ x+1, & \text{if } x \in (-2,-1) \\ \frac{1}{x}, & \text{if } x \in [-1,0) \end{cases}$.

The second fundamental result of this paper is:

***Theorem 5.3**: For every $b \in \mathbf{Q}$, there are two laws of internal composition, let's say „ ♥” and „ ♠”, on the set $\mathbf{Q}_{b,+\infty}$ such that ($\mathbf{Q}_{b,+\infty}$, ♥, ♠) to become is a (commutative) field isomorphic to the field ($\mathbf{Q}$,+,·).*

Proof: We transfer the ring structure from **Q** to $\mathbf{Q}_{b,+\infty}$, using the bijection function:

$g_b : \mathbf{Q} \to \mathbf{Q}_{b,+\infty}$, where, $g_b(x)=\begin{cases} b-\frac{1}{x}, & \text{if } x \in (-\infty,-1] \\ b-x+1, & \text{if } x \in (-1,0) \\ b+x+2, & \text{if } x \in [0,+\infty) \end{cases}$,

and

$g_b^{-1} : \mathbf{Q}_{b,+\infty} \to \mathbf{Q}$, is defined by: $g_b^{-1}(x)=\begin{cases} \frac{1}{b-x}, & \text{if } x \in (b,b+1] \\ b-x+1, & \text{if } x \in (b+1,b+2) \\ x-b-2, & \text{if } x \in [b+2,+\infty) \end{cases}$.

Hence, according to Vălcan (2017), obtain the two composition laws „♥” and „♠” on $\mathbf{Q}_{b,+\infty}$. Let be x, y$\in\mathbf{Q}_{b,+\infty}$. For defining the law „♥”, we distinguish the following cases:

<u>Case 1</u>: x, y∈(b,b+1]. Then:

$$x♥y=g_b(g_b^{-1}(x)+g_b^{-1}(y))=g_b\left(-\frac{1}{x-b}-\frac{1}{y-b}\right)=b+\frac{(x-b)\cdot(y-b)}{(x-b)+(y-b)}.$$

<u>Case 2</u>: x∈(b,b+1] and y∈(b+1,b+2). Then:

$$x♥y=g_b(g_b^{-1}(x)+g_b^{-1}(y))=g_b\left(-\frac{1}{x-b}+b-y+1\right)$$

$$=\begin{cases} b+\frac{x-b}{(x-b)[(y-b)-1]+1}, & \text{if } (x-b)\cdot[(y-b)-2]\geq -1 \\ b+\frac{1}{x-b}+(y-b), & \text{if } \begin{cases}(x-b)\cdot[(y-b)-1]>-1 \\ (x-b)\cdot[(y-b)-2]<-1\end{cases} \end{cases}.$$

<u>Case 3</u>: x∈(b,b+1] și y∈[b+2,+∞). Then:

$$x♥y=g_b(g_b^{-1}(x)+g_b^{-1}(y))=g_b\left(-\frac{1}{x-b}+y-b-2\right)$$

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$$=\begin{cases} b+\dfrac{x-b}{1-(x-b)\cdot[(y-b)-2]}, & \text{if } (x-b)\cdot[(y-b)+1]\geq 1 \\ b-\dfrac{1}{b-x}-(y-b)+3, & \text{if } \begin{cases}(x-b)\cdot[(y-b)-2]>1\\(x-b)\cdot[(y-b)-1]<1\end{cases} \\ b-\dfrac{1}{x-b}+(y-b), & \text{if } (x-b)\cdot[(y-b)-2]\leq 1 \end{cases}.$$

Case 4: x, y∈(b+1,b+2). Then:

$$x♥y=g_a(g_b^{-1}(x)+g_b^{-1}(y))=g_b((b-x+1)+(b-y+1))$$

$$=\begin{cases} b+(x-b)+(y-b)-1, & \text{if } 3\leq (x-b)+(y-b)<4 \\ b+\dfrac{1}{(x-b)+(y-b)-2}, & \text{if } 2<(x-b)+(y-b)<3 \end{cases}.$$

Case 5: x∈(b+1,b+2) and y∈[b+2,+∞). Then:

$$x♥y=g_b(g_b^{-1}(x)+g_b^{-1}(y))=g_b(b-x+1+y-b-2)=g_b[-(x-b)+(y-b)-1]$$

$$=\begin{cases} b+(x-b)-(y-b)+2, & \text{if } \begin{cases}y>x\\y<x+1\end{cases} \\ b-(x-b)+(y-b)+1, & \text{if } y\geq x+1 \end{cases}.$$

Case 6: x, y∈[b+2,+∞). Then:

$$x♥y=g_b(g_b^{-1}(x)+g_b^{-1}(y))=g_b[(x-b)-2+(y-b)-2]=b+(x-b)+(y-b)-2.$$

Therefore, for every x, y∈$\mathbf{Q}_{b,+\infty}$,

$$x♥y=b+\frac{(x-b)\cdot(y-b)}{(x-b)+(y-b)}, \text{ if } x, y\in(b,b+1];$$

$$x♥y=\begin{cases} b+\dfrac{x-b}{(x-b)[(y-b)-1]+1}, & \text{if } (x-b)\cdot[(y-b)-2]\geq -1 \\ b+\dfrac{1}{x-b}+(y-b), & \text{if } \begin{cases}(x-b)\cdot[(y-b)-1]>-1\\(x-b)\cdot[(y-b)-2]<-1\end{cases} \end{cases}, \text{ if } x\in(b,b+1] \text{ and } y\in(b+1,b+2);$$

$$x♥y=\begin{cases} b+\dfrac{x-b}{1-(x-b)\cdot[(y-b)-2]}, & \text{if } (x-b)\cdot[(y-b)+1]\geq 1 \\ b-\dfrac{1}{b-x}-(y-b)+3, & \text{if } \begin{cases}(x-b)\cdot[(y-b)-2]>1\\(x-b)\cdot[(y-b)-1]<1\end{cases} \\ b-\dfrac{1}{x-b}+(y-b), & \text{if } (x-b)\cdot[(y-b)-2]\leq 1 \end{cases} \text{ and } x\in(b,b+1] \text{ and } y\in[b+2,+\infty);$$

$$x♥y=\begin{cases} b+(x-b)+(y-b)-1, & \text{if } 3\leq (x-b)+(y-b)<4 \\ b+\dfrac{1}{(x-b)+(y-b)-2}, & \text{if } 2<(x-b)+(y-b)<3 \end{cases} \text{ and } x, y\in(b+1,b+2);$$

$$x♥y=\begin{cases} b+(x-b)-(y-b)+2, & \text{dacă } \begin{cases}y>x\\y<x+1\end{cases} \\ b-(x-b)+(y-b)+1, & \text{dacă } y\geq x+1 \end{cases}, \text{ if } x\in(b+1,b+2) \text{ and } y\in[b+2,+\infty);$$

$$x♥y=b+(x-b)+(y-b)-2, \text{ if } x, y\in[b+2,+\infty).$$

Now, defining the law „♠”, we distinguish the following cases:

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Case 1: x, y∈(b,b+1]. Then:

$$x♠y=g_b(g_b^{-1}(x)\cdot g_b^{-1}(y))=g_b\left(\frac{1}{x-b}\cdot\frac{1}{y-b}\right)=b+\frac{1}{(x-b)\cdot(y-b)}+2.$$

Case 2: x∈(b,b+1] și y∈(b+1,b+2). Then:

$$x♠y=g_b(g_b^{-1}(x)\cdot g_b^{-1}(y))=g_b\left(-\frac{1}{x-b}\cdot(b-y+1)\right)=b+\frac{(y-b)-1}{x-b}+2.$$

Case 3: x∈(b,b+1] și y∈[b+2,+∞). Then:

$$x♠y=g_b(g_b^{-1}(x)\cdot g_b^{-1}(y))=g_b\left(-\frac{1}{x-b}\cdot(y-b-2)\right)=\begin{cases}b+\frac{x-b}{(y-b)-2}, & \text{if } y\geq x+2\\ b+\frac{(y-b)-2}{x-b}+1, & \text{if } y<x+2\end{cases}.$$

Case 4: x, y∈(b+1,b+2). Then:

$$x♠y=g_a(g_b^{-1}(x)\cdot g_b^{-1}(y))=g_b((b-x+1)\cdot(b-y+1))=g_b((1+(b-x))\cdot(1+(b-y))$$

$$=b+(b-x)\cdot(b-y)+(b-x)+(b-y)+3.$$

Case 5: x∈(b+1,b+2) și y∈[b+2,+∞). Then:

$$x♠y=g_b(g_b^{-1}(x)\cdot g_b^{-1}(y))=g_b((b-x+1)\cdot(y-b-2))=g_b([1-(x-b)]\cdot[(y-b)-2])$$

$$=\begin{cases}b+\frac{1}{[(x-b)-1]\cdot[(y-2)-2]}, & \text{if } [(x-b)-1]\cdot[(y-b)-2]\geq 1\\ b+(x-b)\cdot(y-b)-2\cdot(x-b)-(y-b)+3, & \text{if } 0<[(x-b)-1]\cdot[(y-b)-2]<1\end{cases}.$$

Case 6: x, y∈[b+2,+∞). Then:

$$x♠y=g_b(g_b^{-1}(x)\cdot g_b^{-1}(y))=g_b([(x-b)-2)]\cdot[(y-b)-2])=b+(x-b)\cdot(y-b)-2\cdot(x-b)-2\cdot(y-b)+6.$$

Therefore, for every x, y∈$\mathbf{Q}_{b,+\infty}$,

$x♠y=b+\frac{1}{(x-b)\cdot(y-b)}+2$, if x, y∈(b,b+1];

$x♠y=b+\frac{(y-b)-1}{x-b}+2$, if x∈(b,b+1] and y∈(b+1,b+2);

$x♠y=\begin{cases}b+\frac{x-b}{(y-b)-2}, & \text{if } y\geq x+2\\ b+\frac{(y-b)-2}{x-b}+1, & \text{if } y<x+2\end{cases}$ and x∈(b,b+1] and y∈[b+2,+∞);

x♠y=b+(b-x)·(b-y)+(b-x)+(b-y)+3, if x, y∈(b+1,b+2);

$x♠y=\begin{cases}b+\frac{1}{[(x-b)-1]\cdot[(y-2)-2]}, & \text{if } [(x-b)-1]\cdot[(y-b)-2]\geq 1\\ b+[(x-b)-1]\cdot[(y-b)-2], & \text{if } [(x-b)-1]\cdot[(y-b)-2]\in(0,1)\end{cases}$, if x∈(b+1,b+2) and y∈[b+2,+∞);

x♠y=b+(x-b)·(y-b)-2·(x-b)-2·(y-b)+6, if x, y∈[b+2,+∞).

On the other hand,

$e_{Q_{b,+\infty}}=g_b(e_{\mathbf{Q}})=g_b(0)=b+2$

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and

$$-x_{Q_{b,+\infty}}=g_b(-g_b^{-1}(x))=\begin{cases} b+\dfrac{1}{x-b}+2 & ,\text{ if } x\in(b,b+1] \\ x+1 & ,\text{ if } x\in(b+1,b+2) \\ b+2 & ,\text{ if } x=b+2 \\ x-1 & ,\text{ if } x\in(b+2,b+3) \\ b-\dfrac{1}{2-(x-b)} & ,\text{ if } x\geq b+3 \end{cases},$$

and:

$$1_{Q_{b,+\infty}}=g_b(1_{\mathbf{Q}})=b+3$$

and

$$x^{-1}_{Q_{b,+\infty}}=g_b\left(\frac{1}{g_b^{-1}(x)}\right)=\begin{cases} x+1 & ,\text{ if } x\in(b,b+1) \\ b+1 & ,\text{ if } x=b+1 \\ b+\dfrac{1}{(x-b)-1}+1 & ,\text{ if } x\in(b+1,b+2) \\ b+\dfrac{1}{(x-b)-2}+2 & ,\text{ if } x\in(b+2,+\infty) \end{cases}.$$

Therefore, according to Vălcan (2017), ($\mathbf{Q}_{b,+\infty}$,♥,♠) is a (commutative) field isomorphic to the field ($\mathbf{Q}$,+,·).

Now let's show that, indeed, the function:

$g_b : \mathbf{Q} \to \mathbf{Q}_{b,+\infty}$, defined by: $$g_b(x)=\begin{cases} b-\dfrac{1}{x} & ,\text{ if } x\in(-\infty,-1] \\ b-x+1, & \text{if } x\in(-1,0) \\ b+x+2, & \text{if } x\in[0,+\infty) \end{cases}$$

is an isomorphism between the two fields. For every x, y∈$\mathbf{Q}$,

$$g_b(x+y)=\begin{cases} b-\dfrac{1}{x+y} & ,\text{ if } x+y\in(-\infty,-1] \\ b-x-y+1, & \text{if } x+y\in(-1,0) \\ b+x+y+2, & \text{if } x+y\in[0,+\infty) \end{cases}$$ and $$g_b(x\cdot y)=\begin{cases} b-\dfrac{1}{x\cdot y} & ,\text{ if } x\cdot y\in(-\infty,-1] \\ b-x\cdot y+1, & \text{if } x\cdot y\in(-1,0) \\ b+x\cdot y+2, & \text{if } x\cdot y\in[0,+\infty) \end{cases}.$$

Now, to determine the expressions $g_b(x)$♥$g_b(y)$, respectively $g_b(x)$♠$g_b(y)$, we distinguish the following cases:

Case 1: x, y∈(-∞,-1]. Then:

$g_b(x)=b-\dfrac{1}{x}\in(b,b+1]$ and $g_b(y)=b-\dfrac{1}{y}\in(b,b+1]$.

It follows that:

$g_b(x)$♥$g_b(y)=b+\dfrac{1}{x+y}$ and $g_b(x)$♠$g_b(y)=b+x\cdot y+2$.

Case 2: x∈(-∞,-1] and y∈(-1,0). Then:

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$g_b(x)=b-\frac{1}{x}\in(b,b+1]$ and $g_b(y)=b-y+1\in(b+1,b+2)$.

It follows that:

$g_b(x)\heartsuit g_b(y)=b-\frac{1}{x+y}$ and $g_b(x)\spadesuit g_b(y)=b+x\cdot y+2$.

Case 3: $x\in(-\infty,-1]$ and $y\in[0,+\infty)$. Then:

$g_b(x)=b-\frac{1}{x}\in(b,b+1]$ and $g_b(y)=b+y+2\in[b+2,+-\infty)$.

It follows that:

$$g_b(x)\heartsuit g_b(y)=\begin{cases} b-\frac{1}{x+y} & ,\text{if } x+y+1\geq 0 \\ b-x-y+1 & ,\text{if } \begin{cases} x+y>0 \\ x+z<-1 \end{cases} \\ b+x+y+2, & \text{if } x+y+1\leq 0 \end{cases}=\begin{cases} b-\frac{1}{x+y} & ,\text{if } x+y+1\geq 0 \\ b+x+y+2, & \text{if } x+y+1\leq 0 \end{cases},$$

because the second option is not possible, and

$$g_b(x)\spadesuit g_b(y)=\begin{cases} b-\frac{1}{x\cdot y} & ,\text{if } x\cdot y\geq -1 \\ b-x\cdot y+1, & \text{if } x\cdot y<-1 \end{cases}.$$

Case 4: $x, y\in(-1,0)$. Then:

$g_b(x)=b-x+1\in(b+1,b+2)$ and $g_b(y)=b-y+1\in(b+1,b+2)$.

It follows that:

$$g_b(x)\heartsuit g_b(y)=\begin{cases} b-x-y+1, & \text{if } -2\leq x+y\leq -1 \\ b-\frac{1}{x+y} & ,\text{if } x+y\in(-1,0) \end{cases}$$ and $g_b(x)\spadesuit g_b(y)=b+x\cdot y+2$.

Case 5: $x\in(-1,0)$ and $y\in[0,+\infty)$. Then:

$g_b(x)=b-x+1\in(b+1,b+2)$ and $g_b(y)=b+y+2\in[b+2,+\infty)$.

It follows that:

$$g_b(x)\heartsuit g_b(y)=\begin{cases} b-x-y+1, & \text{if } x+y\in(-2,0) \\ b+x+y+2, & \text{if } x+y\geq 0 \end{cases}$$ and $$g_b(x)\spadesuit g_b(y)=\begin{cases} b-\frac{1}{x\cdot y} & ,\text{if } x\cdot y\leq -1 \\ b-x\cdot y+1, & \text{if } x\cdot y\in(-1,0) \end{cases}.$$

Case 6: $x, y\in[0,+\infty)$. Then:

$g_b(x)=b+x+2\in[b+2,+-\infty)$ and $g_b(y)=b+y+2\in[b+2,+-\infty)$.

It follows that:

$g_b(x)\heartsuit g_b(y)=b+x+y+2$ and $g_b(x)\spadesuit g_b(y)=b+x\cdot y+2$.

Hence, for every $x, y\in\mathbf{Q}$,

$g_b(x+y)=g_b(x)\heartsuit g_b(y)$ and $g_b(x\cdot y)=g_b(x)\spadesuit g_b(y)$.

Now, we can say that the theorem is completely proved.

At the end of this paragraph, three further remarks are required:

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Remark 5.4: For b=0 we get the structure of commutative field on the set $\mathbf{Q}_{0,+\infty}$, of positive rationals numbers, transferred from the field ($\mathbf{Q}$,+,$\cdot$) by function:

$$g_0 : \mathbf{Q} \to \mathbf{Q}_{0,+\infty}, \qquad \textit{defined by}: \qquad f_0(x)=\begin{cases} -\frac{1}{x} & \text{, if } x \in (-\infty,-1] \\ -x+1 & \text{, if } x \in (-1,0) \\ x+2 & \text{, if } x \in [0,+\infty) \end{cases},$$

whose inverse is the function:

$$g_0^{-1} : \mathbf{Q}_{0,+\infty} \to \mathbf{Q}, \qquad \textit{defined by}: \qquad g_0^{-1}(x)=\begin{cases} -\frac{1}{x} & \text{, if } x \in (b,b+1] \\ -x+1 & \text{, if } x \in (b+1,b+2) \\ x-2 & \text{, if } x \in [b+2,+\infty) \end{cases}.$$

Remark 5.5: As demonstrated above, for any a, b$\in\mathbf{Q}$, the fields ($\mathbf{Q}_{-\infty,a}$,♣,♦) and ($\mathbf{Q}_{b,+\infty}$,♥,♠) are commutative, and the diagram (A) is a commutative diagram of commutative fields.

Remark 5.6: For every number a$\in\mathbf{Q}$, there are two laws of internal composition, let's say „♣" and „♦", on the set $\mathbf{Q}_{-\infty,a}$, such that ($\mathbf{Q}_{-\infty,a}$,♣,♦) to become is a commutative field isomorphic to the field ($\mathbf{Q}$,+,$\cdot$) and there are two laws of internal composition, let's say „♥" and „♠", on the set $\mathbf{Q}_{a,+\infty}$, such that ($\mathbf{Q}_{a,+\infty}$,♥,♠) to become is a commutative field isomorphic to the field ($\mathbf{Q}$,+,$\cdot$) and so that the following diagram (B), in Figure 2, is a commutative diagram of commutative field:

$$\begin{tikzcd} \mathbf{Q} \arrow[r, "f_a"] \arrow[d, "g_a"'] & \mathbf{Q}_{-\infty,a} \arrow[dl, "h_{a,a}=g_a\circ f_a^{-1}."] \\ \mathbf{Q}_{a,+\infty} & \end{tikzcd}$$

Figure 2. Diagram (B)

In the diagram (B);

$\mathbf{Q}_{-\infty,a}=\{x\in\mathbf{Q} \mid x<a\}$ and $\mathbf{Q}_{a,+\infty}=\{x\in\mathbf{Q} \mid x>a\}$.

and functions:

$f_a : \mathbf{Q} \to \mathbf{Q}_{-\infty,a}$ and $g_a : \mathbf{Q} \to \mathbf{Q}_{a,+\infty}$

defined by:

$$f_a(x)=\begin{cases} a+\frac{1}{x} & \text{, if } x \in (-\infty,-1] \\ a+x-1 & \text{, if } x \in (-1,0) \\ a-x-2 & \text{, if } x \in [0,+\infty) \end{cases} \qquad \text{and} \qquad g_a(x)=\begin{cases} a-\frac{1}{x} & \text{, if } x \in (-\infty,-1] \\ a-x+1 & \text{, if } x \in (-1,0) \\ a+x+2 & \text{, if } x \in [0,+\infty) \end{cases},$$

are bijections (they are precisely isomorphisms of fields), whose inverses are:

$f_a^{-1} : \mathbf{Q}_{-\infty,a} \to \mathbf{Q}$ and $g_a^{-1} : \mathbf{Q}_{a,+\infty} \to \mathbf{Q}$,

defined by:

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$$f_a^{-1}(x)=\begin{cases} a-x-2, \text{ if } x\in(-\infty,a-2] \\ x-a+1, \text{ if } x\in(a-2,a-1) \\ \dfrac{1}{x-a} \quad, \text{ if } x\in[a-1,a) \end{cases} \quad \text{and} \quad g_a^{-1}(x)=\begin{cases} \dfrac{1}{a-x} \quad, \text{ if } x\in(a,a+1] \\ a-x+1, \text{ if } x\in(a+1,a+2) \\ x-a-2, \text{ if } x\in[a+2,+\infty) \end{cases}.$$

The function that achieves the isomorphism between the rings ($\mathbf{Q}_{-\infty,a}$,♣,♦) and ($\mathbf{Q}_{a,+\infty}$,♥,♠) is:

$h_{a,a}=g_a\circ f_a^{-1}\ \mathbf{Q}_{-\infty,a}: \to \mathbf{Q}_{a,+\infty}$;

where, for every $x\in\mathbf{Q}_{-\infty,a}$,

$$h_{a,a}(x)=g_a\circ f_a^{-1}(x)=g_a(f_a^{-1}(x))=\begin{cases} a-\dfrac{1}{f_a^{-1}(x)} \quad, \text{ if } f_a^{-1}(x)\in(-\infty,-1] \\ a-f_a^{-1}(x)+1, \text{ if } f_a^{-1}(x)\in(-1,0) \\ a+f_a^{-1}(x)+2, \text{ if } f_a^{-1}(x)\in[0,+\infty) \end{cases}$$

$$=\begin{cases} a-x+a \quad, \text{ if } x\in[a-1,a) \\ a-x+a-1+1 \ , \text{ if } x\in(a-2,a-1) \\ a+a-x-2+2, \text{ if } x\in(-\infty,a-2] \end{cases}=2\cdot a\text{-}x=a+(a\text{-}x),$$

and

$h_{a,a}^{-1}=f_a\circ g_a^{-1}:\mathbf{Q}_{a,+\infty}\to\mathbf{Q}_{-\infty,a}$;

where, for every $x\in\mathbf{Q}_{-\infty,a}$,

$h_{a,a}^{-1}(x)=f_a(g_a^{-1}(x))=a\text{-}(x\text{-}a)$.

5. Findings

Therefore, we answered the two questions in Paragraph 3. Thus, for any number a, $b\in\mathbf{Q}$ there are two pairs of laws of internal composition on the sets $\mathbf{Q}_{-\infty,a}$ and $\mathbf{Q}_{b,+\infty}$, let's say „♣" and „♦", respectively „♥" and „♠", so that ($\mathbf{Q}_{-\infty,a}$,♣,♦) and ($\mathbf{Q}_{b,+\infty}$,♥,♠) become commutative fields isomorphic to the field ($\mathbf{Q}$,+,·).

Concretely, on the set of rationals smaller than 3, $\mathbf{Q}_{-\infty,3}$, and on the set of rationals greater than 5, $\mathbf{Q}_{5,+\infty}$, we can define two pairs of laws of internal composition so that let's say „♣" and „♦", respectively „♥" and „♠", so that ($\mathbf{Q}_{-\infty,3}$,♣,♦) and ($\mathbf{Q}_{5,+\infty}$,♥,♠) become commutative fields isomorphic to the field ($\mathbf{Q}$,+,·).

6. Conclusion

As a general conclusion, we can say that any problem / exercise or equation in the set of rational numbers (**Q**,+,·) can be solved in either of these two sets ($\mathbf{Q}_{-\infty,a}$,♣,♦), respectively ($\mathbf{Q}_{b,+\infty}$,♥,♠); but also vice versa, i.e. any problem / exercise or equation in any of these two sets ($\mathbf{Q}_{-\infty,a}$,♣,♦), respectively ($\mathbf{Q}_{b,+\infty}$,♥,♠), we can solve it in the set of rational numbers ($\mathbf{Q}$,+,·). Of course, after solving them in the new set, the solution is interpreted in the original set.

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Moreover, because these fields ($\mathbf{Q}_{-\infty,a}$,♣,♦) and ($\mathbf{Q}_{b,+\infty}$,♥,♠) are isomorphic to the field **Q**, but also isomorphic to each other, it follows that these new fields have their own subsets / rings of "*integer*" ($\mathbf{Z}_{-\infty,a}$,♣,♦) and ($\mathbf{Z}_{b,+\infty}$,♥,♠)) and "*natural*" ($\mathbf{N}_{-\infty,a}$,♣,♦) and ($\mathbf{N}_{b,+\infty}$,♥,♠) numbers respectively , isomorphic to each other, and isomorphic to the sets (**Z**,+,·) and (**N**,+,·) respectively.

Of course, this paper is one of Didactics of Mathematics and is addressed to pupils, students or teachers attentive and interested in these issues, which we believe we have formed, in this way, a good image about these matters.

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